Electromagnetic Theory
Prof. Michael J. Ruiz, UNC Asheville (doctorphys on YouTube)
Chapter A Notes. Vector Analysis

A0. Vector

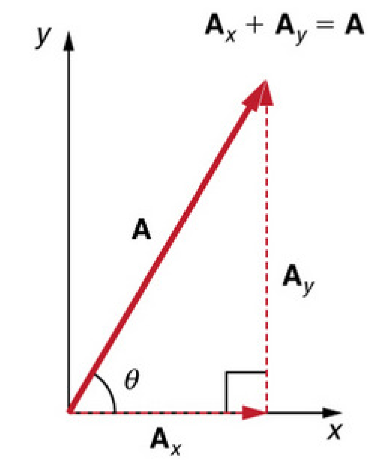
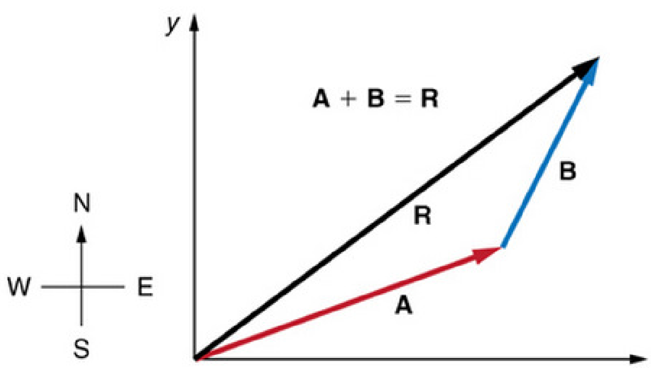


Figure Courtesy OpenStax College. Vector Addition and Subtraction: Analytical Methods, Connexions Website. http://cnx.org/content/m42128/1.10/, June 20, 2012.

$$\vec{A} = A_x \hat{i} + A_y \hat{j}$$

$$\vec{A}_x = A_x \hat{i} \quad \text{and} \quad \vec{A}_y = A_y \hat{j}$$

A1. Vector Addition and Subtraction

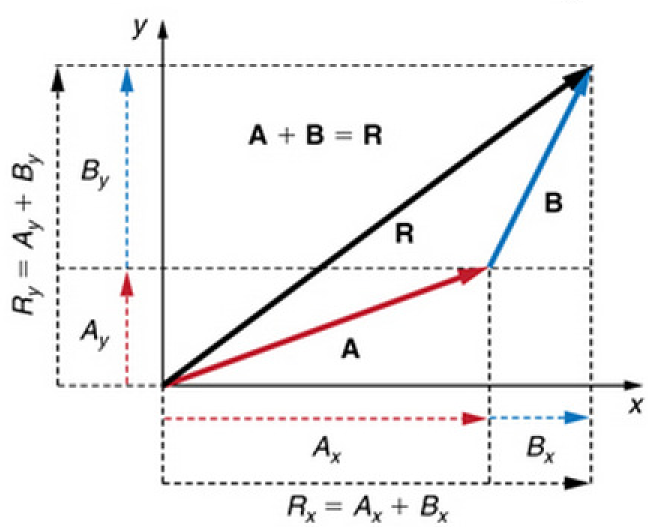


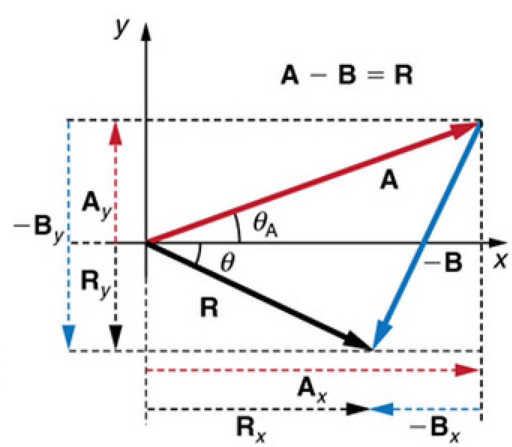
Figures Courtesy OpenStax College

$$\vec{A} = A_x \hat{i} + A_y \hat{j}$$

$$\vec{B} = B_x \hat{i} + B_y \hat{j}$$

$$\vec{R} = (A_x + B_x)\hat{i} + (A_y + B_y)\hat{j}$$





The Resultant: $\vec{R} = \vec{A} + \vec{B} = (A_x + B_x)\,\hat{i} + (A_y + B_y)\,\hat{j} = R_x\,\hat{i} + R_y\,\hat{j}$

Adding the Negative Vector (Subtraction): $\vec{A} - \vec{B} = (A_x - B_x)\,\hat{i} + (A_y - B_y)\,\hat{j}$

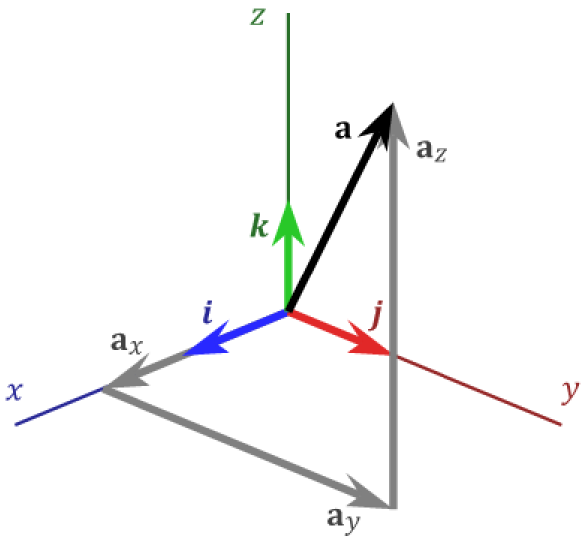


Image Courtesy Acdx, Wikipedia

Examples of vectors in three dimensions.

$$\vec{a} = a_x\,\hat{i} + a_y\,\hat{j} + a_z\,\hat{k}$$

$$\vec{A} = A_x\,\hat{i} + A_y\,\hat{j} + A_z\,\hat{k}$$

$$\vec{F} = F_x\,\hat{i} + F_y\,\hat{j} + F_z\,\hat{k}$$

A2. Scalar Multiplication

$$\alpha\vec{A} = \alpha(A_x\,\hat{i} + A_y\,\hat{j} + A_z\,\hat{k})$$

$$\alpha\vec{A} = \alpha A_x\,\hat{i} + \alpha A_y\,\hat{j} + \alpha A_z\,\hat{k}$$

A3. Dot Product

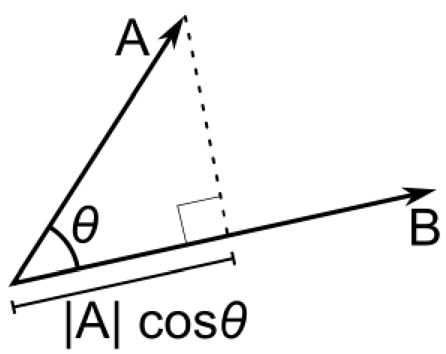


Image from Wikimedia Commons

$$\vec{A} \cdot \vec{B} = AB\cos\theta$$

Note: $\hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = (1)(1)\cos 0° = 1$

$$\hat{i} \cdot \hat{j} = \hat{i} \cdot \hat{k} = \hat{j} \cdot \hat{k} = (1)(1)\cos 90° = 0$$

Also note $\vec{B} \cdot \vec{A} = BA\cos\theta = AB\cos\theta = \vec{A} \cdot \vec{B}$

Using the rules for the dot product of the unit vectors we arrive at the following.

$$\vec{A} \cdot \vec{B} = (A_x \hat{i} + A_y \hat{j} + A_z \hat{k}) \cdot (B_x \hat{i} + B_y \hat{j} + B_z \hat{k})$$

$$\vec{A} \cdot \vec{B} = A_x B_x \hat{i} \cdot \hat{i} + A_x B_y \hat{i} \cdot \hat{j} + A_x B_z \hat{i} \cdot \hat{k}$$

$$+ A_y B_x \hat{j} \cdot \hat{i} + A_y B_y \hat{j} \cdot \hat{j} + A_y B_z \hat{j} \cdot \hat{k}$$

$$+ A_z B_x \hat{k} \cdot \hat{i} + A_z B_y \hat{k} \cdot \hat{j} + A_z B_z \hat{k} \cdot \hat{k}$$

$$\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z$$

More notation: $\hat{e}_1 = \hat{i}$, $\hat{e}_2 = \hat{j}$, and $\hat{e}_3 = \hat{k}$. Then $\vec{A} = A_1 \hat{e}_1 + A_2 \hat{e}_2 + A_3 \hat{e}_3$.

The unit vectors are also called basis vectors and we use subscripts that can take on values 1, 2, and 3. The dot product of two arbitrary unit vectors can then be written as

$$\hat{e}_i \cdot \hat{e}_j = \delta_{ij} \text{ where } \delta_{ij} = 1 \text{ if } i = j \text{ and } \delta_{ij} = 0 \text{ if } i \neq j.$$

Here is more new notation: $\vec{A} = \sum_{i=1}^{3} A_i \hat{e}_i$ and $\vec{B} = \sum_{i=1}^{3} B_i \hat{e}_i$

$$\vec{A} \cdot \vec{B} = \sum_{i=1}^{3} A_i \hat{e}_i \cdot \sum_{j=1}^{3} B_j \hat{e}_j = \sum_{i=1}^{3} \sum_{j=1}^{3} A_i B_j \hat{e}_i \cdot \hat{e}_j = \sum_{i=1}^{3} \sum_{j=1}^{3} A_i B_j \delta_{ij} = \sum_{i=1}^{3} A_i B_i$$

$$\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z \text{ or } \vec{A} \cdot \vec{B} = A_1 B_1 + A_2 B_2 + A_3 B_3$$

Einstein Summation Convention: $\vec{A} = A_i \hat{e}_i$ and $\vec{B} = B_i \hat{e}_i$

$$\vec{A} \cdot \vec{B} = A_i \hat{e}_i \cdot B_j \hat{e}_j = A_i B_j \hat{e}_i \cdot \hat{e}_j = A_i B_j \delta_{ij} = A_i B_i$$

Leopold Kronecker (1823-1891)
Courtesy School of Mathematics and Statistics
University of St. Andrews, Scotland

The Kronecker Delta symbol is defined as

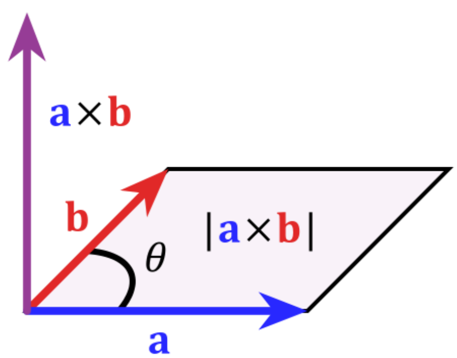
$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$$

and named after the German mathematician Leopold Kronecker. It is a symmetric symbol.

PA1 (Practice Problem). Find the angle theta between the two vectors using the two dot product definitions. Check your answer with a graphical diagram.

$$\vec{A} = \hat{i} - \hat{j} \text{ and } \vec{B} = \hat{i} + \hat{j} .$$

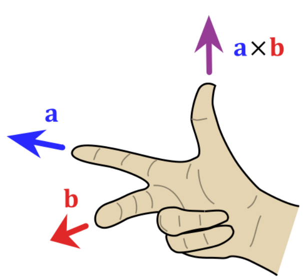
A4. Cross Product



Cross Product (Images Courtesy Acdx, Wikipedia)

$$\vec{A} \times \vec{B} = AB \sin\theta \, \hat{n} \, , \; \vec{a} \times \vec{b} = ab \sin\theta \, \hat{n}$$

where the unit vector $\hat{n}$ is perpendicular to the plane formed by $\vec{a}$ and $\vec{b}$, according to the right-hand rule as shown in the lower figure.



Or you can use the "right-hand screwdriver rule" where you get under the plane and apply the screwdriver to turn "a" into "b" advancing along "n". By the way ab sinθ is the area shown in the parallelogram.

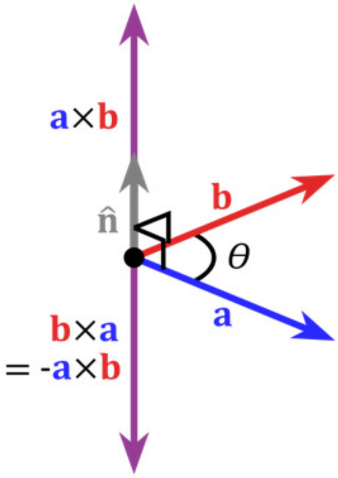


Image Courtesy Acdx, Wikipedia

Note that if you flip the order of the vectors, you get a vector in the opposite direction according to the right-hand rule.

$$\vec{b}\times\vec{a} = ba\sin\theta(-\hat{n})$$

$$\vec{b}\times\vec{a} = -\vec{a}\times\vec{b} \quad \text{and} \quad \vec{B}\times\vec{A} = -\vec{A}\times\vec{B}$$

The right hand-rule with the unit vectors gives us these relations below.

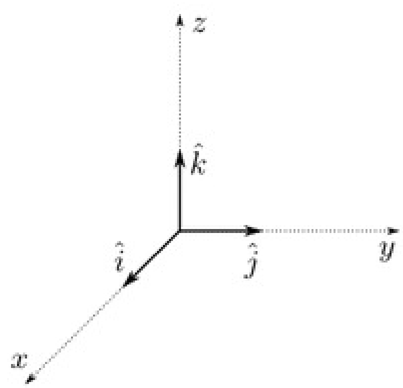


Image Courtesy Acdx, Wikipedia

$$\hat{i}\times\hat{j} = \hat{k} \qquad \hat{j}\times\hat{i} = -\hat{k} \qquad \hat{i}\times\hat{i} = 0$$

$$\hat{j}\times\hat{k} = \hat{i} \qquad \hat{k}\times\hat{j} = -\hat{i} \qquad \hat{j}\times\hat{j} = 0$$

$$\hat{k}\times\hat{i} = \hat{j} \qquad \hat{i}\times\hat{k} = -\hat{j} \qquad \hat{k}\times\hat{k} = 0$$

We now apply these rules.

$\vec{A}\times\vec{B} = (A_x\hat{i} + A_y\hat{j} + A_z\hat{k})\times(B_x\hat{i} + B_y\hat{j} + B_z\hat{k})$ works out to:

$$\vec{A}\times\vec{B} = A_xB_x\,\hat{i}\times\hat{i} + A_xB_y\,\hat{i}\times\hat{j} + A_xB_z\,\hat{i}\times\hat{k}$$

$$+A_yB_x\,\hat{j}\times\hat{i} + A_yB_y\,\hat{j}\times\hat{j} + A_yB_z\,\hat{j}\times\hat{k}$$

$$+A_zB_x\,\hat{k}\times\hat{i} + A_zB_y\,\hat{k}\times\hat{j} + A_zB_z\,\hat{k}\times\hat{k}$$

$$\vec{A}\times\vec{B} = A_xB_x\cdot 0 + A_xB_y\,\hat{k} + A_xB_z(-\hat{j})$$

$$+A_yB_x(-\hat{k}) + A_yB_y\cdot 0 + A_yB_z\,\hat{i}$$

$$+A_zB_x\,\hat{j} + A_zB_y(-\hat{i}) + A_zB_z\cdot 0$$

$$\vec{A}\times\vec{B} = (A_y B_z - A_z B_y)\,\hat{i} + (A_z B_x - A_x B_z)\,\hat{j} + (A_x B_y - A_y B_x)\,\hat{k}$$

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$= \hat{i}(A_y B_z - A_z B_y) - \hat{j}(A_x B_z - A_z B_x) + \hat{k}(A_x B_y - A_y B_x)$$

We now switch to our index notation. where $\hat{e}_1 = \hat{i}$, $\hat{e}_2 = \hat{j}$, and $\hat{e}_3 = \hat{k}$.

The cross-product rules can be summarized by writing

$$\hat{e}_i \times \hat{e}_j = \varepsilon_{ijk}\, \hat{e}_k \quad \text{where}$$

$$\varepsilon_{ijk} = \begin{cases} +1 & \text{if } (i,j,k) \text{ is } (1,2,3),(3,1,2) \text{ or } (2,3,1), \\ -1 & \text{if } (i,j,k) \text{ is } (1,3,2),(3,2,1) \text{ or } (2,1,3), \\ 0 & \text{if } i=j \text{ or } j=k \text{ or } k=i \end{cases}$$

Tullio Levi-Civita (1873-1941)
Courtesy School of Mathematics and Statistics
University of St. Andrews, Scotland

The symbol ε_{ijk} is called the Levi-Civita or permutation symbol. It is an antisymmetric symbol. If you swap any two indices you introduce a minus sign. If any two indices are the same you get zero.

$$\vec{A}\times\vec{B} = \sum_{i=1}^{3} A_i\, \hat{e}_i \times \sum_{j=1}^{3} B_j\, \hat{e}_j = \sum_{i=1}^{3}\sum_{j=1}^{3} A_i B_j\, \hat{e}_i \times \hat{e}_j$$

$$\vec{A}\times\vec{B} = \sum_{i=1}^{3}\sum_{j=1}^{3} A_i B_j \varepsilon_{ijk}\, \hat{e}_k$$

The same with Einstein's summation convention is:

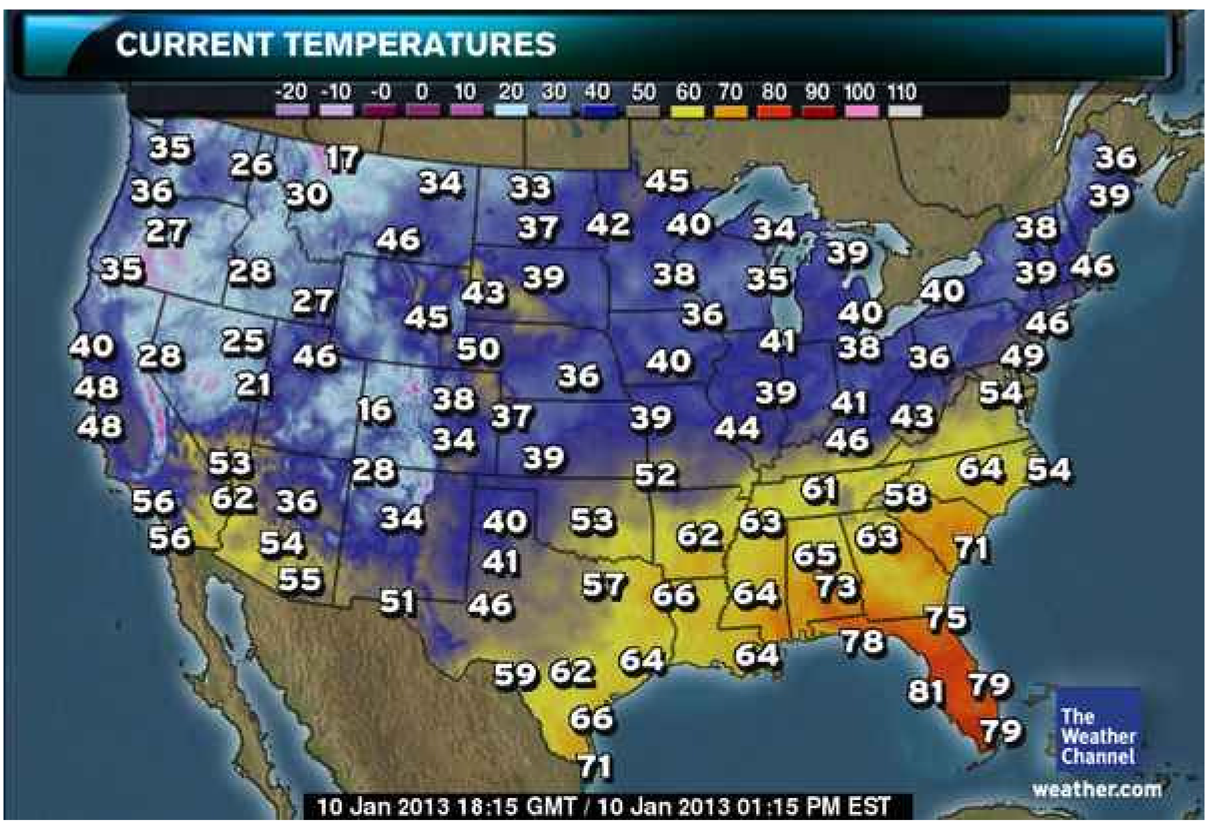
$$\vec{A}\times\vec{B} = A_i\,\hat{e}_i\times B_j\,\hat{e}_j = A_i B_j \varepsilon_{ijk}\,\hat{e}_k$$

PA2 (Practice Problem). Find the angle theta between the two vectors using the two cross product definitions. Check your answer against PA1.

$$\vec{A} = \hat{i} - \hat{j} \text{ and } \vec{B} = \hat{i} + \hat{j} \, .$$

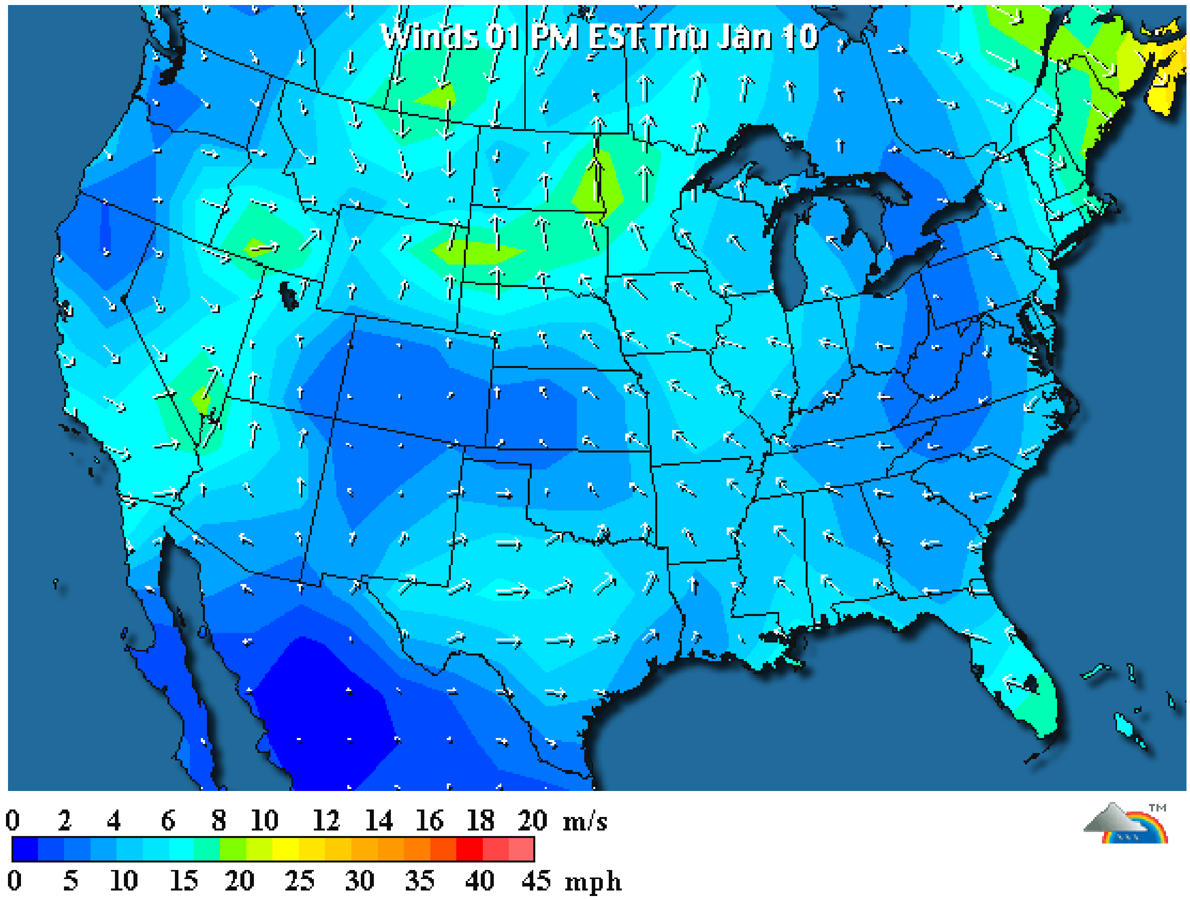
A5. Tensors

Tensor of Rank 0. This is your scalar. A single number is all you need. There is no directional vector or anything like that. The length of a vector stripped of its direction is a scalar. Another example is temperature at each point in a room: T = T(x,y,z) or you can add the time variable so the temperatures change in time. Below is a snapshot of the temperatures across the United States at the time of the writing of this chapter.



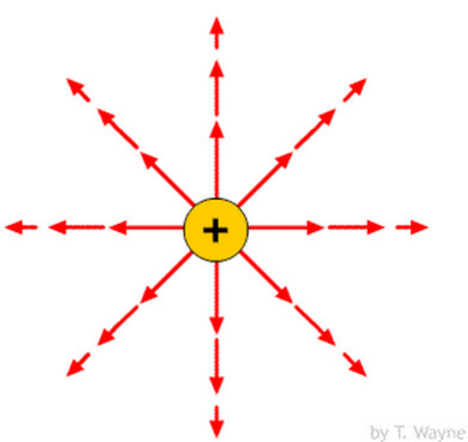
Courtesy The Weather Channel

Tensor of Rank 1. This is your vector. It has magnitude and direction. It can also be a function of the spatial coordinates as well as time.



Courtesy Weather Underground, Inc.

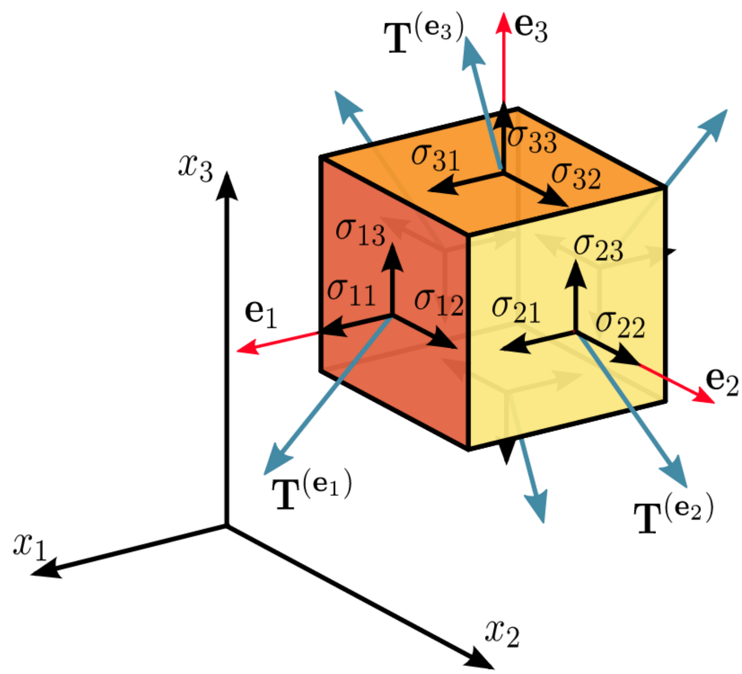
Wind velocity has magnitude (the speed) and direction. The length of the vector arrows indicate the magnitude of the velocity and the arrow points in the direction of the wind. Technically, speed is a scalar, the magnitude. When you promote speed to a vector you add the direction. However, often velocity is used informally for just speed.



Charge Image Courtesy Tony Wayne

Here is a vector field produced by a plus charge. Note the symmetry as all vectors points outward away from the positive charge. Also note that the lengths of the vectors decrease as you get farther away from the charge. The strength weakens according to the inverse square law. In contrast to the weather case this field has a simple formula.

Tensor of Rank 2. Among friends, you can think of a tensor of rank 2 as needing 3 x 3 = 9 components.



Courtesy Sanpaz, Wikipedia

The stress tensor is an example. We need to consider the force on each of the three main faces defined by the three unit vectors. On each surface there is a normal force and two shear (sideway) forces.

We need not consider all 6 faces since mechanical equilibrium guarantees that there will be opposing forces and torques on the opposite sides.

This means we need 9 quantities to define the stress.

Matrix notation will assist us here. For the tensors of Rank 0, 1, and 2 respectively, we can write for three dimensional space.

$$s = [T] = T \qquad \vec{A} = \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} \qquad \sigma_{ij} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$$

For two dimensions we have for tensors of Rank 0, 1, and 2 respectively listed below.

$$s = [T] = T \qquad \vec{A} = \begin{bmatrix} A_x \\ A_y \end{bmatrix} \qquad M_{ij} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$$

But not all matrices are tensors. There are transformation properties that need to be satisfied. However, if you need vector components as in the stress analysis, then you are on good grounds that you are probably dealing with a tensor.

Tensor of Rank 3. What would this be? How about a Tensor of Rank n in m dimensions?