

Electromagnetic Theory
Prof. Ruiz, UNC Asheville, doctorphys on YouTube
Chapter C Notes. Gauss's Law

C1. The Electric Force

Classical Physics around 1800 included the following main laws of physics. These are Newton's Second Law, Newton's Law of Universal Gravitation, and Coulomb's Law.

$$F = ma \qquad F_G = \frac{GMm}{r^2} \qquad F_E = \frac{kQq}{r^2}$$

With gravity (our second equation), masses attract according to an inverse-square law. Similarly with the electric force law (our third equation), the force between charges satisfies an inverse-square law, but there you can have repulsion as well as attraction: "likes" (same sign for the charges) repel and "unlikes" (a plus charge and a minus charge) attract.

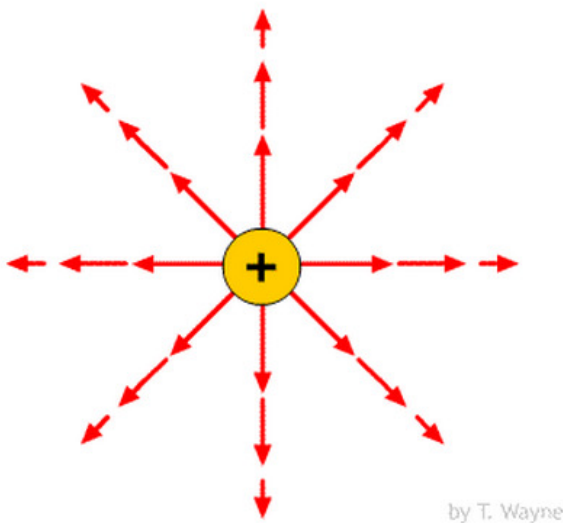
In the meter-kilogram-second (MKS) system, the electric constant $k = \frac{1}{4\pi\epsilon_0}$, where

ϵ_0 is called the permittivity of free space. We can define a gravitational field $g = \frac{GM}{r^2}$ for a point mass M and write the force that a test mass m experiences as $F_G = mg$. Following along these lines, we define the electric $E = \frac{kQ}{r^2}$ field for a

point charge Q and the force on a test charge q is then $F_E = qE$. The force is outward from the point charge so we

include a unit vector \hat{r} that points outward from the point particles in each case.

Charge Image Courtesy Tony Wayne



$$\vec{E} = \frac{kQ}{r^2} \hat{r}$$

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{Q}{r^2} \hat{r}$$

The electric field due to a group of charges q_i at distances r_i is found by adding the vector electric fields. The little unit vectors \hat{r}_i are in all kinds of directions depending on the line of sight to each charge from our position. For n point charges we have

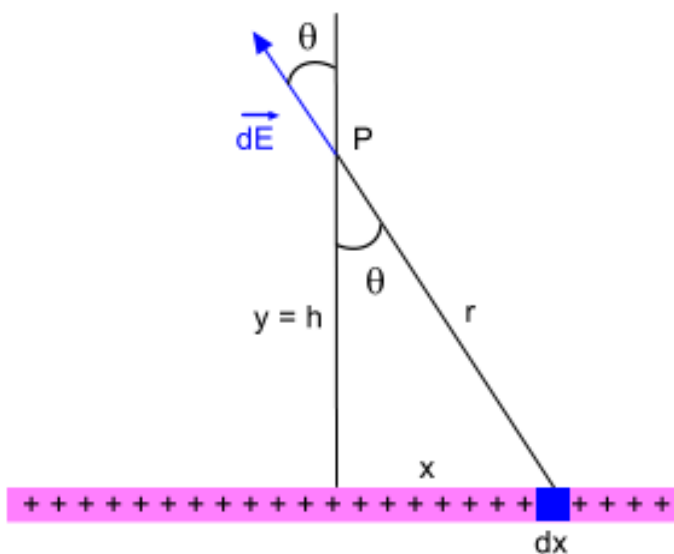
$$\vec{E} = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^n \frac{q_i}{r_i^2} \hat{r}_i.$$

For a continuous distribution with charge density ρ the electric field is given by an integral.

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \int \frac{dq}{r^2} \hat{r} = \frac{1}{4\pi\epsilon_0} \int_V \frac{\rho dV}{r^2} \hat{r}$$

You integrate over the volume of charge and have to keep track of the unit vector constantly shifting directions.

C2. The "Midpoint" Electric Field for a Line of Charge



Take the length of the line of charge to be L and the point P of interest to be $y = h$ from the center of the line. Let the linear charge density be $\lambda = Q/L$, where the total charge is Q .

Due to the symmetry, the x component of the electric field will cancel with the contribution from the charge at $-x$ on the other side. So our net electric field is in the y direction.

$$E = E_y = \int \cos \theta \frac{1}{4\pi\epsilon_0} \frac{dq}{r^2} \quad \text{where} \quad dq = \lambda dx$$

$$E = \int_{-L/2}^{L/2} \frac{h}{\sqrt{x^2 + h^2}} \frac{1}{4\pi\epsilon_0} \frac{\lambda dx}{(x^2 + h^2)} = 2 \int_0^{L/2} \frac{\lambda h}{4\pi\epsilon_0} \frac{dx}{(x^2 + h^2)^{3/2}}$$

$$E = \frac{\lambda h}{2\pi\epsilon_0} \int_0^{L/2} \frac{dx}{(x^2 + h^2)^{3/2}}$$

We can look the integral up in the tables of integrals. But a theoretical physicist is likely to play with this and try to figure it out. The basic structure of this integral is

$$I = \int \frac{dx}{(1+x^2)^{3/2}}.$$

Trig substitutions do not seem promising due to the 3/2 power and plus sign. However, a clever trig method will work. But here, we will expand in a power series using the binomial expansion formula.

$$(1+z)^n = 1 + nz + \frac{n(n-1)}{1 \cdot 2} z^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} z^3 + \dots$$

$$(1+x^2)^{-3/2} = 1 + \left(-\frac{3}{2}\right)x^2 + \frac{\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{1 \cdot 2} x^4 + \frac{\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(-\frac{7}{2}\right)}{1 \cdot 2 \cdot 3} x^6 + \dots$$

We now integrate term by term.

$$x + \left(-\frac{3}{2}\right) \frac{x^3}{3} + \frac{\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{1 \cdot 2} \frac{x^5}{5} + \frac{\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(-\frac{7}{2}\right)}{1 \cdot 2 \cdot 3} \frac{x^7}{7} + \dots$$

$$x + \left(-\frac{1}{2}\right)x^3 + \frac{\left(-\frac{3}{2}\right)\left(-\frac{1}{2}\right)}{1 \cdot 2} x^5 + \frac{\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(-\frac{1}{2}\right)}{1 \cdot 2 \cdot 3} x^7 + \dots$$

$$x \left[1 + \left(-\frac{1}{2}\right)x^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{1 \cdot 2} x^4 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{1 \cdot 2 \cdot 3} x^6 + \dots \right]$$

$$I = x(1+x^2)^{-1/2} = x / \sqrt{1+x^2}$$

Summary: $E = \frac{\lambda h}{2\pi\epsilon_0} \int_0^{L/2} \frac{dx}{(x^2 + h^2)^{3/2}}$ and $I = \int \frac{dx}{(1+x^2)^{3/2}} = \frac{x}{\sqrt{1+x^2}}$

PC1 (Practice Problem). Show that

$$I = \int \frac{dx}{(h^2 + x^2)^{3/2}} = \frac{1}{h^2} \frac{x}{\sqrt{h^2 + x^2}}.$$

Using the result from the practice problem we can continue.

$$E = \frac{\lambda h}{2\pi\epsilon_0} \int_0^{L/2} \frac{dx}{(x^2 + h^2)^{3/2}} = \frac{\lambda h}{2\pi\epsilon_0} \frac{1}{h^2} \frac{x}{\sqrt{x^2 + h^2}} \Bigg|_{x=0}^{x=L/2}$$

$$E = \frac{\lambda h}{2\pi\epsilon_0} \frac{1}{h^2} \left[\frac{L/2}{\sqrt{(L/2)^2 + h^2}} - 0 \right]$$

$$\vec{E} = \frac{\lambda}{2\pi\epsilon_0} \frac{1}{h} \frac{L}{\sqrt{L^2 + 4h^2}} \hat{j} \quad \text{or} \quad \vec{E} = \frac{Q}{2\pi\epsilon_0 h} \frac{1}{\sqrt{L^2 + 4h^2}} \hat{j}$$

Limiting cases are extremely important in physics so we consider shrinking our line down to a point to see if we get the point charge result. Then we consider the infinite line of charge.

Use the expression with the Q explicitly shown since charge stays finite.

$$\lim_{L \rightarrow 0} \vec{E} = \frac{Q}{2\pi\epsilon_0 h} \frac{1}{\sqrt{4h^2}} \hat{j} = \frac{Q}{2\pi\epsilon_0 h} \frac{1}{2h} \hat{j} = \frac{Q}{4\pi\epsilon_0} \frac{1}{h^2} \hat{j}$$

Use the expression with the λ explicitly shown since the charge density stays finite.

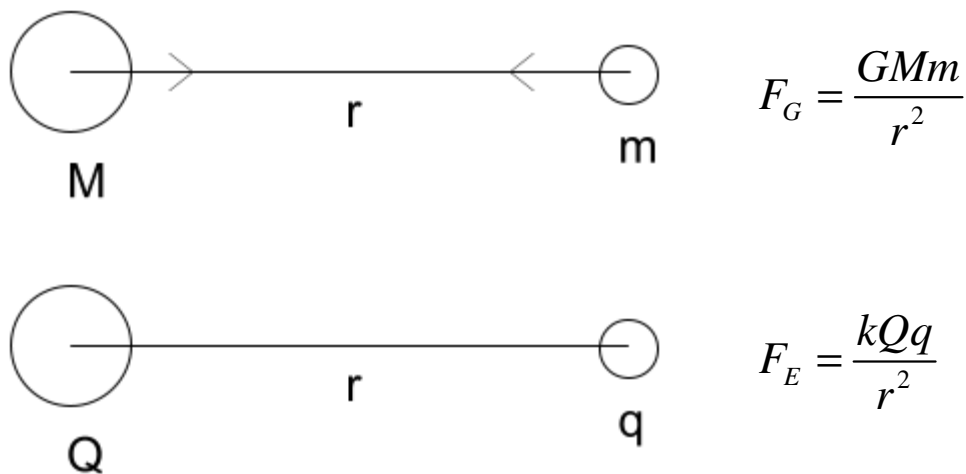
$$\lim_{L \rightarrow \infty} \vec{E} = \sum_{\lim_{L \rightarrow \infty}} \frac{\lambda}{2\pi\epsilon_0} \frac{1}{h} \frac{L}{\sqrt{L^2 + 4h^2}} \hat{j} = \frac{\lambda}{2\pi\epsilon_0} \frac{1}{h} \hat{j}$$

C3. Gauss's Law

Gauss's Law is another way of expressing Coulomb's Law. Coulomb's Law is

$$F_E = \frac{kQq}{r^2} \text{ where the constant } k = \frac{1}{4\pi\epsilon_0}$$

in the MKS system. Note the similarity with the form of Newton's Law of Universal Gravitation below. We are giving the magnitudes below.

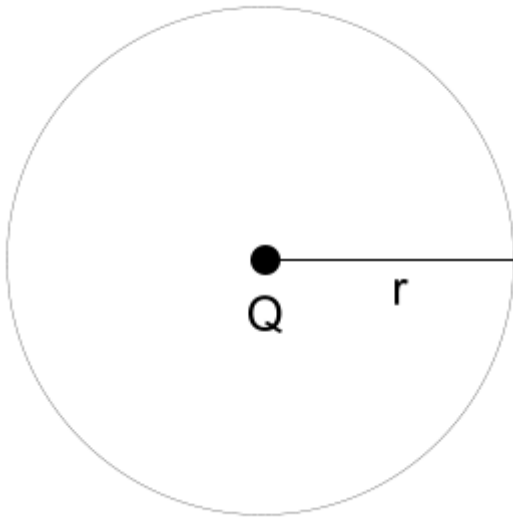


Both are inverse-square laws. The Q represents charge, replacing the M which represents mass. The r is the distance between the centers of each mass or charge. Sometimes you see a minus sign in front of the gravity equation to remind us that the force is attractive. With the charges, when they are opposite in sign, you get the attraction.

Think about these properties called mass and charge. A fundamental force in nature means we endow matter with a property that goes with the force. For gravity it is mass M (or m). For the electric force, it is charge Q (or q). If you have mass, you experience gravity. If you have charge, you experience the electric force.

We define force fields for gravitation and the electric force by taking the smaller mass m to be 1 and the smaller charge q to be 1. Then we have forces per unit mass or charge:

$$g = \frac{GM}{r^2} \qquad E = \frac{kQ}{r^2}$$



Consider a charge Q at the origin and make a sphere at distance r to surround this charge.

The electric field at a distance r from the charge is

$$E = \frac{kQ}{r^2}$$

Since our force is a vector and will point outward for a positive test charge q , we write this in vector form as

$$\vec{E} = \frac{kQ}{r^2} \hat{r}$$

The \hat{r} vector is a unit vector pointing away from the charge Q . Its precise direction depends on where you are on the sphere. This is unlike your unit vectors in Cartesian coordinates, \hat{i} , \hat{j} , and \hat{k} , which always point in the same directions.

We proceed to define a differential patch of area on the sphere and give it a unit vector direction outward. This is common practice with areas, i.e., to define area orientations with unit vectors perpendicular to the surfaces. When you come to think of this, that is the most conventional way to tell someone how to orient a plane piece of paper - by a unit vector perpendicular to the paper for the direction you want.

$$d\vec{A} = \hat{r} dA$$

Don't worry about the actual details for $d\vec{A}$ as we will not need explicit expressions. We will be talking at the most fundamental level for the most part.

We want to take $\vec{E} \cdot d\vec{A}$ and integrate over all the area on the sphere. When we integrate over a closed area we include a nice loop to emphasize that our area encloses on itself:

$$\oiint \vec{E} \cdot d\vec{A}$$

Let's do this integral. We have $\vec{E} = \frac{kQ}{r^2} \hat{r}$ and $d\vec{A} = \hat{r} dA$. Then,

$$\oiint \vec{E} \cdot d\vec{A} = \oiint \frac{kQ}{r^2} \hat{r} \cdot \hat{r} dA.$$

The dot product $\hat{r} \cdot \hat{r}$ is equal to 1 since a dot product of any unit vector with itself is 1. The dot product between two vectors is the multiplication of the magnitudes times the cosine of the angle between them. The angle between a vector and itself is zero.

$$\oiint \vec{E} \cdot d\vec{A} = \oiint \frac{kQ}{r^2} dA$$

Since we have a sphere here with a radius r that does not change and we want the surface area, we can pull out the constants:

$$\oiint \vec{E} \cdot d\vec{A} = \frac{kQ}{r^2} \oiint dA.$$

This might still look scary but the integral is simply the surface area of a sphere. You know this. It is $A = 4\pi r^2$. So we get

$$\oiint \vec{E} \cdot d\vec{A} = \frac{kQ}{r^2} 4\pi r^2 = 4\pi kQ.$$

Note that our constant $k = \frac{1}{4\pi\epsilon_0}$. So we wind up with

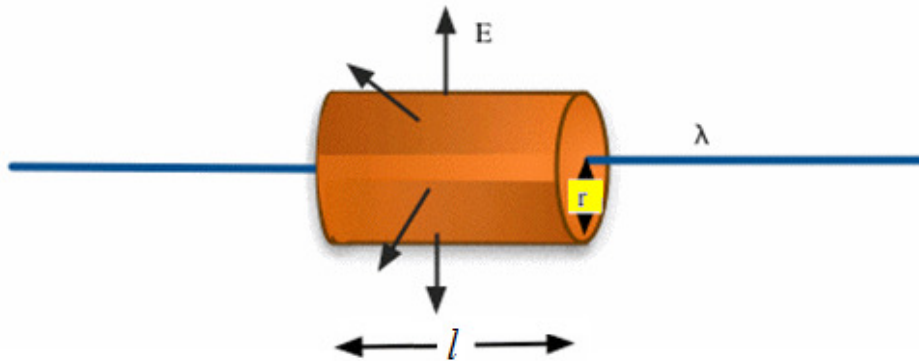
$$\oiint \vec{E} \cdot d\vec{A} = 4\pi kQ = 4\pi \frac{1}{4\pi\epsilon_0} Q = \frac{Q}{\epsilon_0}$$

This form is called Gauss's Law. It is also the first Maxwell equation. Sometimes authors write in or inside as a subscript for Q to emphasize it is the charge inside the enclosure.

$$\oiint \vec{E} \cdot d\vec{A} = \frac{Q}{\epsilon_0} \quad \text{or} \quad \oiint \vec{E} \cdot d\vec{A} = \frac{Q_{in}}{\epsilon_0}$$

Always remember that Q is the charge inside the enclosure. This is an important point. We will see how to use this formula, including this idea below, which you saw in your intro physics course on electricity and magnetism (E&M). That would have been the second semester course of the one-year-long sequence.

Here is how you can apply this powerful formula to calculate the electric field due to an infinite line of charge that has charge density λ , i.e., charge per unit length.



Courtesy Department of Physics, Carleton College

You sketch an enclosed surface, a cylinder in this case, so that the electric field lines pierce the surface at 90° . Then for a distance r from the line, you apply Gauss's law. Since the E fields pierce the cylindrical part, we only worry about that part of the surface area. The dot product there will give 1 for the cosine while for the flat left and right sides the dot product will give zero.

$$\oiint \vec{E} \cdot d\vec{A} = \frac{Q_{in}}{\epsilon_0} \text{ becomes } E(2\pi r l) = \frac{\lambda l}{\epsilon_0}, \text{ which gives}$$

$$E = \frac{1}{2\pi r \epsilon_0} \lambda \text{ or } E = \frac{\lambda}{2\pi \epsilon_0} \frac{1}{r}$$

Note that the L dropped out. Pretty neat, right? A $1/r$ rule instead of inverse square since the line of charge is infinite and reinforces the field. This agrees with our earlier result, which was a longer calculation. Note that $h = r$.

$$\lim_{L \rightarrow \infty} \vec{E} = \lim_{L \rightarrow \infty} \frac{\lambda}{2\pi \epsilon_0} \frac{1}{h} \frac{L}{\sqrt{L^2 + 4h^2}} \hat{j} = \frac{\lambda}{2\pi \epsilon_0} \frac{1}{h} \hat{j}$$

PC2 (Practice Problem). Find the electric field a distance z above an infinite plane of charge with uniform surface density σ .