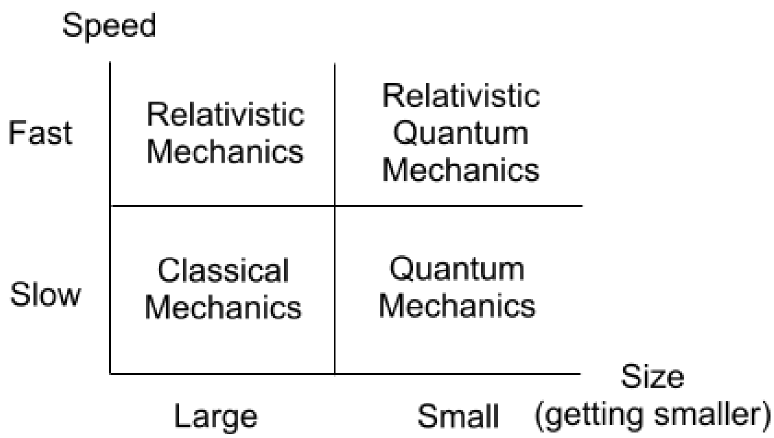
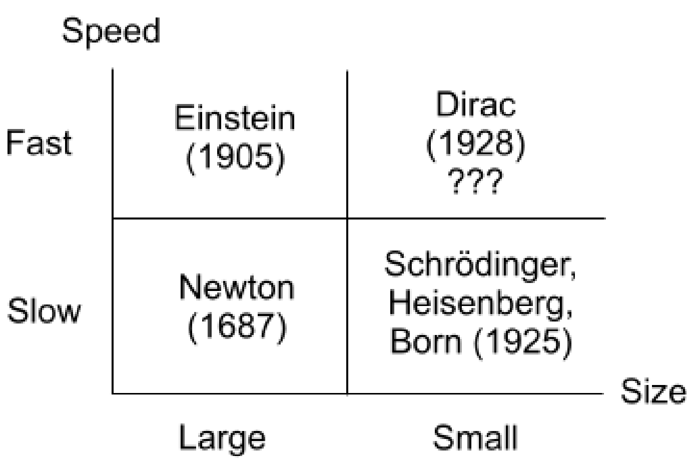
Electromagnetic Theory
Prof. Ruiz, UNC Asheville, doctorphys on YouTube
Chapter D Notes. The Magnetic Force

D0. The Four Realms of Physics



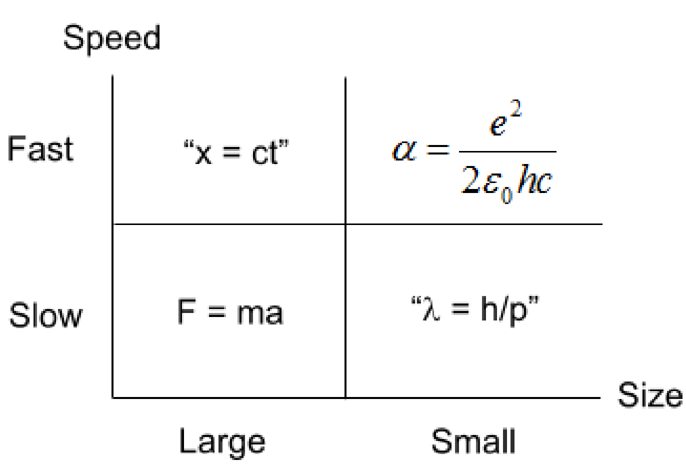
In the figures are the four realms of physics according to speed and size.

The main contributors in each of the realms are given in the second figure. The physics in the fast-small realm is not finished. Dirac started it with his Dirac equation in 1928.



Feynman, Tomonaga, and Schwinger made the next big breakthrough in the 1940s with their independent work in quantum electrodynamics, i.e., QED.

The last figure gives a sample equation from each of the realms. The α is the fine structure constant. It is dimensionless, which indicates a connection among h, c, and electromagnetic theory. Equations in the fast-small have both h and c.



Modern Physics

"x = ct"

"λ = h/p"

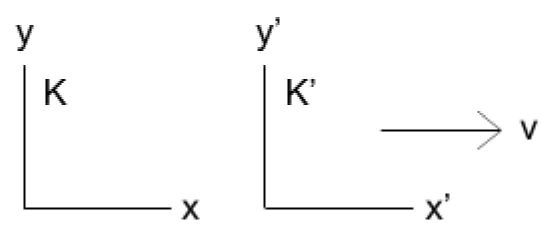
PD1 (Practice Problem). Show that the fine-structure constant $\alpha = \frac{e^2}{2\varepsilon_0 hc}$ is dimensionless using dimensional analysis. Then show that $\alpha = \frac{1}{137}$. The fine structure constant is sometimes written as $\alpha = \frac{e^2}{4\pi\varepsilon_0 \hbar c}$ where $\hbar = \frac{h}{2\pi}$. Therefore, you can write $\alpha = \frac{ke^2}{\hbar c}$. In the cgs (centimeter-gram-second) units, the electric constant is taken to be 1 such that the fine structure constant takes the form $\alpha = \frac{e^2}{\hbar c}$.

D1. Space and Time

Electricity and magnetism is often included with classical physics. However, it is unique in that it needs no corrections to be in agreement with special relativity. In this chapter we will illustrate this by deriving the magnetic field from the electric field and special relativity. Before we begin we need to do some more relativity.

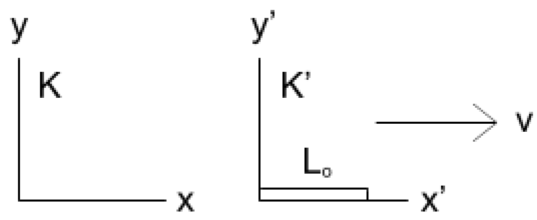
We will pick up from where we left off last time with special relativity - our Lorentz transformation.

The Lorentz transformation between the frames K and K' is given below.



$$x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}} \quad \text{and} \quad t' = \frac{t - \frac{vx}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}.$$

1) Lorentz Contraction. Place a rod of length L_o in the K' frame. What is the length as measured from the K frame?



We start with

$$x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}}$$ and take deltas.

$$\Delta x' = \frac{\Delta x - v\Delta t}{\sqrt{1 - \frac{v^2}{c^2}}}$$

We have $\Delta x' = L_o$ and $\Delta x = L$, the second being the measurement as the rod zips by us. When we make our measurement we want to "nail" the ends at the same instant. Therefore, our measurement takes place such that $\Delta t = 0$. We then have

$$L_o = \frac{L}{\sqrt{1 - \frac{v^2}{c^2}}} \text{ and } L = L_o\sqrt{1 - \frac{v^2}{c^2}}.$$

We measure the moving rod to be contracted. Note that the proper length of the rod is L_o, the length as measured in the frame in which the rod is at rest.

2) Time Dilation. Place a clock in the K' frame. We need to keep the clock still in that frame. What is the time as measured from the K frame? Since the clock is at rest in the K' frame, $\Delta x' = 0$. The proper time T_o is time kept by this stationary clock in its own K' frame. Therefore, $T_o = \Delta t'$. The time $T = \Delta t$ is the time measured in the K frame as the clock zips by. We need to use the equation of the Lorentz transformation that has x', t', and t, so we can find deal with $\Delta x'$, $\Delta t'$, and Δt. We do not see one since we have

$$x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}} \text{ and } t' = \frac{t - x\frac{v}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}.$$

So we write the inverse transformation, i.e., how to get the K coordinates from the K' coordinates. This is easy. You just let v go to minus v. You put yourself in the K' frame and watch the K frame go the other way.

$$x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}} \quad \text{and} \quad t' = \frac{t - x\frac{v}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}} \quad \text{then become}$$

$$x = \frac{x' + vt'}{\sqrt{1 - \frac{v^2}{c^2}}} \quad \text{and} \quad t = \frac{t' + x'\frac{v}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}} .$$

Now we can obtain an equation with $\Delta x'$, $\Delta t'$, and Δt. We use the last one and take deltas.

$$\Delta t = \frac{\Delta t' + \Delta x'\frac{v}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}} , \quad \text{which becomes} \quad T = \frac{T_0 + 0 \cdot \frac{v}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}} .$$

The result is time dilation, a stretching of time:

$$T = \frac{T_0}{\sqrt{1 - \frac{v^2}{c^2}}} .$$

Summary:

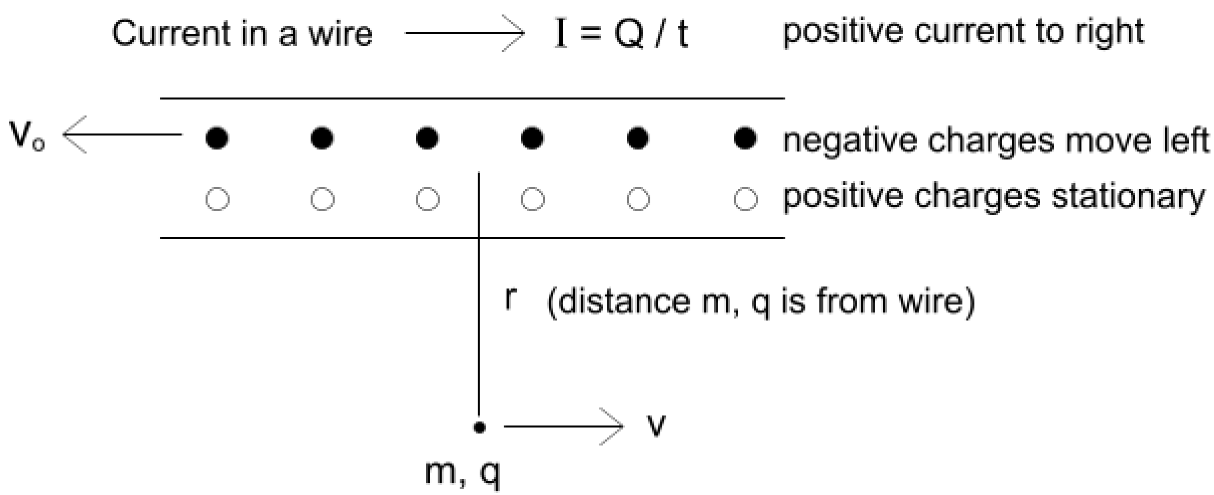
Lorentz Contraction of the moving rod: $L = L_o\sqrt{1 - \frac{v^2}{c^2}}$ as we measure it go by.

Time Dilation of the moving clock: $T = \frac{T_0}{\sqrt{1 - \frac{v^2}{c^2}}}$ as we watch is go by. Our time is longer.

D2. The Magnetic Field

I encountered a form of what you are going to see here in Edward M. Purcell, *Electricity and Magnetism Berkeley Physics Course - Volume 2* (New York, McGraw-Hill, 1965). The usual approach is to take both positive and negative charges moving opposite each other and exploit the symmetry. But the realistic case is to have the electrons move and not the positive ionic cores of the atoms in the metal.

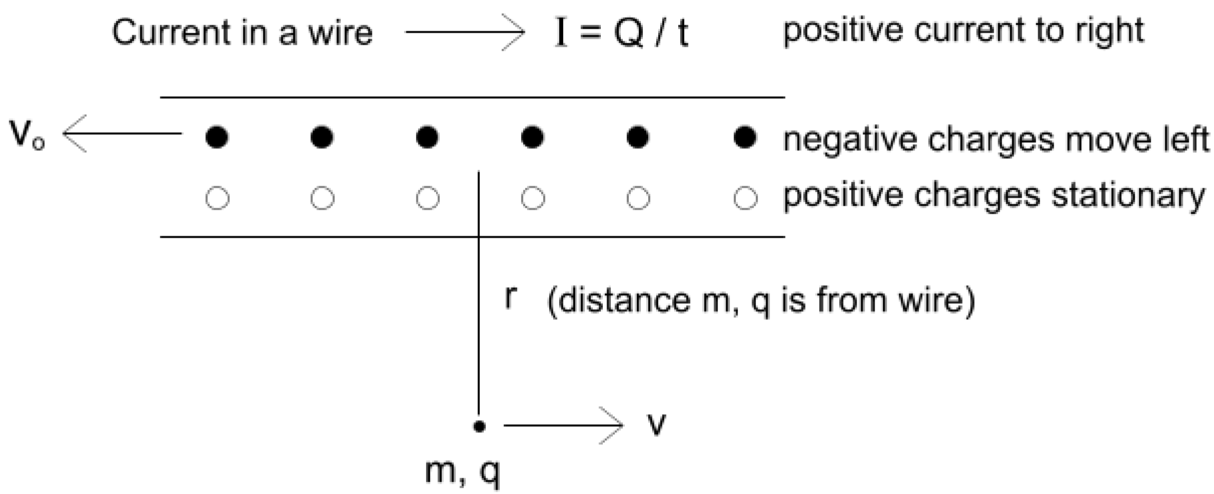
The black dots below are electrons moving to the left in a wire. We make things ideal and consider each electron moving to the left with equal spacing L_{lab}. Think of each electron as moving on its own. The wire is neutral and our charge feels no force if at rest. But if the charge q moves, then the Lorentz contraction effect kicks in. The electrons are spaced closer to each other than the protons, as viewed from the moving frame, and the charge experiences a sideways force attracted to the electrons.



This is the idea. We did it. The rest is algebra. In the above diagram, current is designated by i and the definition for current is charge per unit time. Current is moving charge. The convention is to take the flow in the positive direction, i.e., opposite to the direction of the moving negative charges. **The charge symbol Q will be always positive. We will place a minus in front when necessary. The same goes for λ.**

Applying our line of charge formula $E = \frac{1}{2\pi r}\frac{\lambda}{\varepsilon_0}$ from the previous chapter, we have two lines of charge and write the magnitude of the electric field in the K' frame as

$$E' = \frac{1}{2\pi r}\frac{\lambda_-' - \lambda_+'}{\varepsilon_0} > 0$$ and the vector form is $\vec{E}' = E'\hat{j}$ (pointing up).



Here are our three frames:

Frame K is the laboratory frame.
Frame K' is the moving frame that goes along with the moving mass m (speed v).
Frame K" is the moving frame that rides along with the moving electrons (speed v_o).

Moving mass m "sees" positive charge (the white circles) moving to the left at $\beta = \frac{v}{c}$

Moving mass m "sees" negative charge (the black dots) moving left at $\beta_T = \frac{v_T}{c}$

The speed β_T is found from the relativistic addition of speeds v and v_0. This result is

$$\beta_T = \frac{\beta_0 + \beta}{1 + \beta_0 \beta} .$$

For our $E' = \frac{1}{2\pi r} \frac{\lambda_-' - \lambda_+'}{\varepsilon_0} > 0$, we apply the generic formula for linear charge density $\lambda = \frac{Q}{L}$. i.e., charge per length. In the lab frame we take $\frac{Q}{L_{lab}}$, where each charge is Q and the spacing is L_{lab}. In the lab frame negative and positive cancel and the wire is neutral as the electrons move to the left in step a distance L_{lab} apart. In the K" frame we take the electron separation distance to be L''.

Therefore for $E' = \frac{1}{2\pi r}\frac{\lambda_-' - \lambda_+'}{\varepsilon_0} > 0$ we need λ_-' and λ_+'. From K' the Lorentz contraction for the positive charges is $L_+' = L_{lab}\sqrt{1-\beta^2}$. For the electrons we use the electron frame length times its contraction: $L_-' = L''\sqrt{1-\beta_T{}^2}$. Then

$$\lambda_-' = \frac{Q}{L''\sqrt{1-\beta_T{}^2}} \text{ and } \lambda_+' = \frac{Q}{L_{lab}\sqrt{1-\beta^2}}$$

For the electric field as seen in the K' frame, i.e., the moving mass, we have

$$E' = \frac{1}{2\pi r\varepsilon_0}\left[\frac{Q}{L''\sqrt{1-\beta_T{}^2}} - \frac{Q}{L_{lab}\sqrt{1-\beta^2}}\right]$$

But we want the laboratory spacing here. We want to get rid of that L'', which is the electron spacing in its own K" frame. From the laboratory perspective, the electron spacing is seen to be contracted as

$$L_{lab} = L''\sqrt{1-\beta_0{}^2}\ .$$

We substitute $\frac{1}{L''} = \frac{\sqrt{1-\beta_0{}^2}}{L_{lab}}$ into our E' equation and obtain

$$E' = \frac{1}{2\pi r\varepsilon_0}\frac{Q}{L_{lab}}\left[\frac{\sqrt{1-\beta_0{}^2}}{\sqrt{1-\beta_T{}^2}} - \frac{1}{\sqrt{1-\beta^2}}\right]$$

This looks complicated but it will simplify. We will rewrite this equation at the top of the next page and continue.

$$E' = \frac{1}{2\pi r\varepsilon_0}\frac{Q}{L_{lab}}\left[\frac{\sqrt{1-\beta_0{}^2}}{\sqrt{1-\beta_T{}^2}} - \frac{1}{\sqrt{1-\beta^2}}\right]$$

PD2 (Practice Problem). Use $\beta_T = \dfrac{\beta_0 + \beta}{1+\beta_0\beta}$ to simplify the above expression to

$$E' = \frac{1}{2\pi r\varepsilon_0}\frac{Q}{L_{lab}\sqrt{1-\beta^2}}\beta_0\beta \ .$$

The mess cleared itself up big time! Amazing!

From

$$E' = \frac{1}{2\pi r\varepsilon_0}\frac{Q}{L_{lab}\sqrt{1-\beta^2}}\beta_0\beta$$

the force in the K' frame is

$$F_y{}' = qE' = \frac{q}{2\pi r\varepsilon_0}\frac{Q}{L_{lab}\sqrt{1-\beta^2}}\beta_0\beta \ .$$

But we want the force expressed in the lab frame. Note that this is a y force. Note also that all our frames of reference move horizontally. Therefore, there is no contraction in the y direction and the y-momentum is the same. Therefore we can write in general

$$F_y = \frac{dp_y}{dt} \quad \text{and} \quad F_y{}' = \frac{dp_y}{dt'}$$

It is the time that is different: $dt = \dfrac{dt'}{\sqrt{1-\beta^2}}$, the time dilation we saw earlier. Since

$dt' = \sqrt{1-\beta^2}\,dt$, we have $F_y{}' = \dfrac{dp_y}{dt'} = \dfrac{1}{\sqrt{1-\beta^2}}\dfrac{dp_y}{dt} = \dfrac{F_y}{\sqrt{1-\beta^2}}$.

Therefore,

$$F_y' = \frac{F_y}{\sqrt{1-\beta^2}} = \frac{q}{2\pi r\varepsilon_0}\frac{Q}{L_{lab}\sqrt{1-\beta^2}}\beta_0\beta$$

and in the lab frame, our $F_y = qE$ is going to be as follows.

$$F_y = \frac{q}{2\pi r\varepsilon_0}\frac{Q}{L_{lab}}\beta_0\beta$$

$$F_y = \frac{q}{2\pi r\varepsilon_0}\frac{Q}{L_{lab}}\frac{v_0}{c}\frac{v}{c}$$

$$F_y = \frac{q}{2\pi r}\left[\frac{1}{\varepsilon_0 c^2}\right]\left[\frac{Q}{L_{lab}}v_0\right]v$$

Now, this grouping $\left[\frac{1}{\varepsilon_0 c^2}\right]$ is defined to be $\frac{1}{\varepsilon_0 c^2} \equiv \mu_0$

$$F_y = \frac{q}{2\pi r}\mu_0\left[\frac{Q}{L_{lab}}v_0\right]v\ .$$

This constant μ_0 is the magnetic permeability or simply the permeability of the vacuum or free space. It is analogous the ε_0, the permittivity of the vacuum or free space.

Remember that v_0 is the speed of the electrons as seen from the lab frame and L_{lab} is the spacing of these electrons as seen in the lab frame. So $v_0 = \frac{L_{lab}}{t}$ where t is the time it takes for an electron to travel the distance L_{lab}. Current i is charge per second, i.e., $I = \frac{Q}{t}$. Since $v_0 = \frac{L_{lab}}{t}$, i.e., $\frac{v_0}{L_{lab}} = \frac{1}{t}$, we have $I = \frac{Q}{L_{lab}}v_0$.

Then, $F_y = \frac{q}{2\pi r}\mu_0\left[\frac{Q}{L_{lab}}v_0\right]v$ becomes $F_y = \frac{q}{2\pi r}\mu_0 I v$.

Rearrange a little to write this as

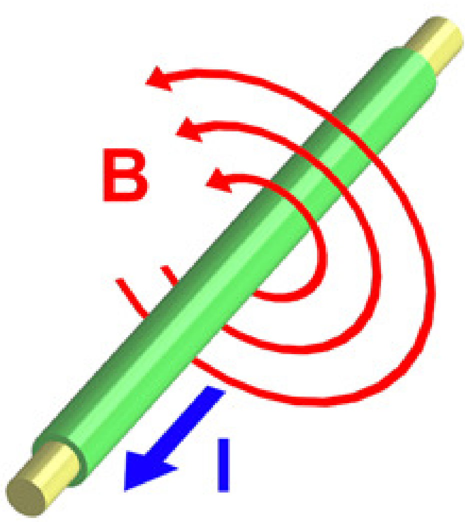
$$F_y = qv\frac{\mu_0 I}{2\pi r}.$$

We have isolated the two parameters q and v that deal with properties of the moving charge. The rest of the pieces relate to the external force field. We call this the magnetic field and designate it with the letter B

$$B = \frac{\mu_0 I}{2\pi r}.$$

The magnetic field is due to the current in the wire. Due to the cylindrical symmetry we can assign the unit vector $\hat{\theta}$. To get a sense of this direction, use your right hand with thumb aligned with the current. The B field then takes on the direction of your curved fingers.

$$\vec{B} = \frac{\mu_0 I}{2\pi r}\hat{\theta}.$$



Here is a figure (Courtesy Wikimedia: Stannered and Wapcaplet) showing the right-hand rule in action to get the direction of the magnetic field.

$$\vec{B} = \frac{\mu_0 I}{2\pi r}\hat{\theta}$$

The magnitude of our force is

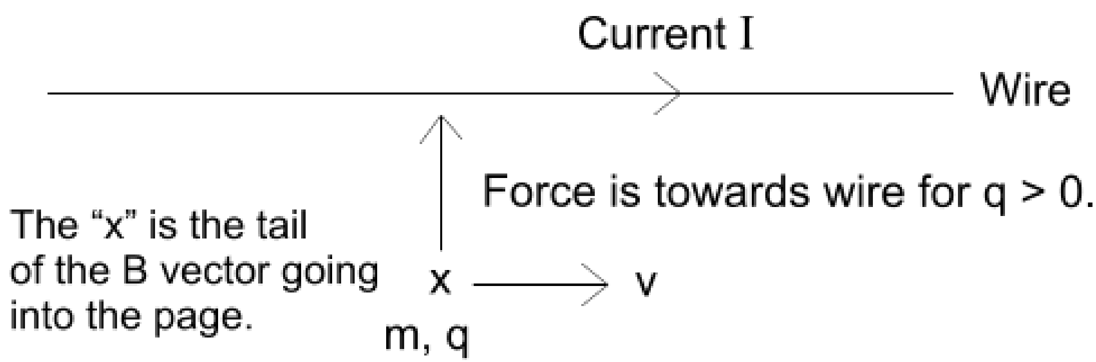
$$F_y = qv\frac{\mu_0 I}{2\pi r} = qvB.$$

Since the force is upward towards the wire we have $\vec{F} = qvB\,\hat{j}$. With $\vec{B} = \frac{\mu_0 I}{2\pi r}\hat{\theta}$ and $\vec{v} = v\,\hat{i}$

along the x direction, we can use a cross product to express the force:

$$\vec{F} = q\,\vec{v}\,x\,\vec{B}\text{, where } \vec{B} = \frac{\mu_0 I}{2\pi r}\hat{\theta}\,.$$

We designate the B vector below with an "x" to represent the tail of the vector pointing into the page. The cross product gets you the force in the right direction.



The general force law which includes both electric and magnetic fields is called the Lorentz force law, named after Lorentz of Lorentz transformation fame.

$$\vec{F} = q\,\vec{E} + q\,\vec{v} \times \vec{B}$$

Physicists like to use the letter i for current and also the square root of minus 1. This can be confusing if you have electrical engineering problems where you need both current and the imaginary number. So electrical engineers define the square root of minus one to be j to avoid confusion in such intricate equations.

D3. Electricity, Relativity, and Magnetism

The deep mystery of the intimate connection among electricity, magnetism, and relativity which we have seen can be summarized in this compact formula.

$$\frac{1}{\varepsilon_0 c^2} \equiv \mu_0$$

The constant of electricity, magnetism, and relativity are related. Light as we will see later is an electromagnetic phenomenon.

$$c^2 = \frac{1}{\varepsilon_0 \mu_0} \quad \text{and} \quad c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$$

In another chapter we will summarize the basic laws of electromagnetic theory in four equations called the Maxwell equations. We will then solve the Maxwell equations in empty space to derive a wave equation for the electric and magnetic fields.

The speed of the waves is given by $c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$. Maxwell discovered this around 1865. When you put in the constants of electricity and magnetism to find out the speed for the waves, you find the speed of light!

Values of constants recommended by CODATA (Committee on Data for Science and Technology, 2010).

$$\varepsilon_0 = 8.854187817 x 10^{-12} \frac{C^2}{N \cdot m^2}$$ ("electric" permittivity of free space)

$$\mu_0 \equiv 4\pi x 10^{-7} \frac{N}{A^2}$$ ("magnetic" permeability of free space)

$$c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} = \frac{1}{\sqrt{(8.854187817 x 10^{-12} \frac{C^2}{N \cdot m^2})(4\pi x 10^{-7} \frac{N}{A^2})}}$$

$$c = 299,792,458 \frac{m}{s}$$

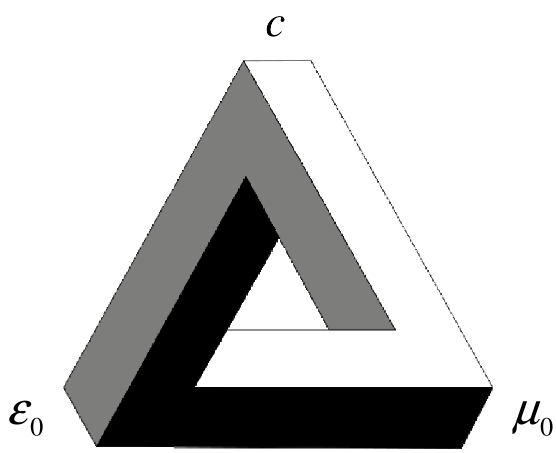
And guess what CODATA lists for the speed of light? See below!

$$c = 299,792,458 \frac{m}{s}$$

This is EXACT. the meter is defined from the speed of light since 1983.

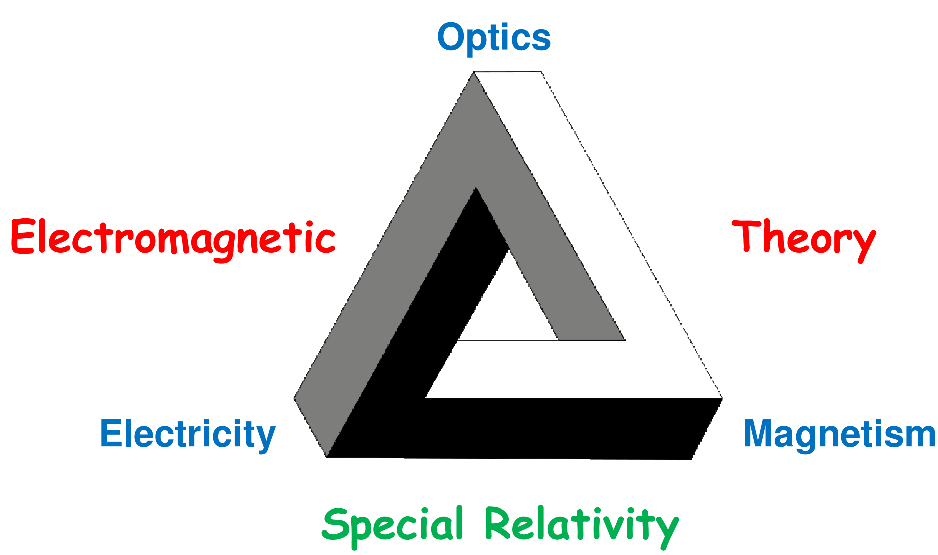
The meter is defined as the distance light travels in $\frac{1}{299,792,458} s$.

We represent the connections with the Impossible Triangle!



Impossible Triangle
Courtesy Tobias R. (Metoc)
Wikimedia Commons

The Impossible Triangle was invented by the Swedish artist Oscar Reutersvärd in the 1930s and independently by the mathematician Roger Penrose in the 1950s. It was also used by M. C. Escher in his works.



Also, the first area of physics to be developed in the realm of the fast and small.

Quantum Electrodynamics (QED)