Electromagnetic Theory
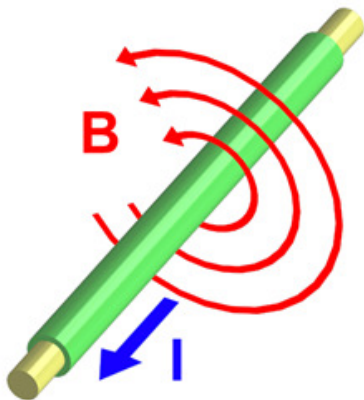**Prof. Ruiz, UNC Asheville, doctorphys on YouTube**
Chapter E Notes. Ampère's Law, Mini Reviews, and Four-Vectors

E1. Ampère's Law (from Intro Physics II and our Result from Class D).

Courtesy Stannered/Wacaplet, Wikimedia, showing the right-hand rule for the direction of the magnetic field.

$$\vec{B} = \frac{\mu_0 I}{2\pi r} \hat{\theta}$$

We can express the magnetic field B as a loop line integral wrapping around the wire. Let's do this backwards. We then arrive at Ampère's Law in integral form.

$$B = \frac{\mu_0 I}{2\pi r}, \qquad B(2\pi r) = \mu_0 I, \qquad B\oint dl = \mu_0 I$$

$$\oint B \, dl = \mu_0 I, \qquad \oint B \, \hat{\theta} \cdot \hat{\theta} \, dl = \mu_0 I, \quad \text{and} \quad \oint \vec{B} \cdot \vec{dl} = \mu_0 I.$$

This is the integral we can use to calculate B for the current in the wire. You did this in your introductory physics class. Since the magnetic field lines wrap around on themselves, if we swallow up a magnet like we did for a charge in Gauss's Law, we get zero. There are no piercings of the magnetic field outward through the surface.

$$\oiint \vec{B} \cdot d\vec{A} = 0$$

This is another Maxwell equation. Here is a summary of what we have so far where we put the surface integrals first. The first is Gauss's Law, an alternate form of Coulomb's Law. The third is Ampère's Law. The second law is not associated with anyone in particular. But it involves the magnetic field and Ampère is a key figure.

$$\oiint \vec{E} \cdot d\vec{A} = \frac{Q}{\varepsilon_0} \qquad \oiint \vec{B} \cdot d\vec{A} = 0 \qquad \oint \vec{B} \cdot \vec{dl} = \mu_0 I$$

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$

The Big Three so Far.

Charles Augustin de Coulomb (1736-1806)	Johann Carl Friedrich Gauss (1777-1855)	André Marie Ampère (1775-1836)

From the School of Mathematics and Statistics, University of St. Andrews, Scotland

E2. Gauss's Law and Ampère's Law (Review from Intro Physics II).

Let's apply Ampère's Law to a thick wire with constant current density. In reality, current tends to flow along the outer surface of the conductor since the electrons repel each other. Imagine a wire cable consisting of a bundle of little wires with thin insulations that we can neglect. We take the radius of the cable to be R and the current density to be J.

We will apply Gauss's Law at the same time to a sphere of constant charge density. In reality charge will move to the surface of a conducting sphere. So we are considering here an ideal material with uniform charge throughout. We take the radius of the sphere to be R and the charge density ρ.

$$\oiint \vec{E} \cdot d\vec{A} = \frac{Q}{\varepsilon_0} \qquad\qquad \oint \vec{B} \cdot \vec{dl} = \mu_0 I$$

$$E(4\pi r^2) = \frac{1}{\varepsilon_0} \rho \frac{4}{3} \pi r^3 = \frac{Q}{\varepsilon_0} \frac{r^3}{R^3} \qquad B(2\pi r) = \mu_0 J \pi r^2 = \mu_0 I \frac{r^2}{R^2}$$

$$E = \frac{Q}{4\pi\varepsilon_0} \frac{r}{R^3} \qquad\qquad B = \frac{\mu_0 I}{2\pi} \frac{r}{R^2}$$

What happens if r = R or r > R?

## E3. Lorentz Force Problems (Review from Intro Physics II).

We will review two problems from your introductory physics course. One will involve the electric field and the other will consider a magnetic field.

Start with the general form for the Lorentz force.

$$\vec{F} = \frac{d\vec{p}}{dt} = q(\vec{E} + \vec{v} \times \vec{B})$$

1) Constant Electric Field (Review from Intro Physics II)

Here there is only an electric field, which is pointing in the positive y direction.

$$\vec{F} = m\frac{d\vec{v}}{dt} = q\vec{E}$$

$$F_x = m\frac{dv_x}{dt} = 0$$

$$F_y = m\frac{dv_y}{dt} = qE$$
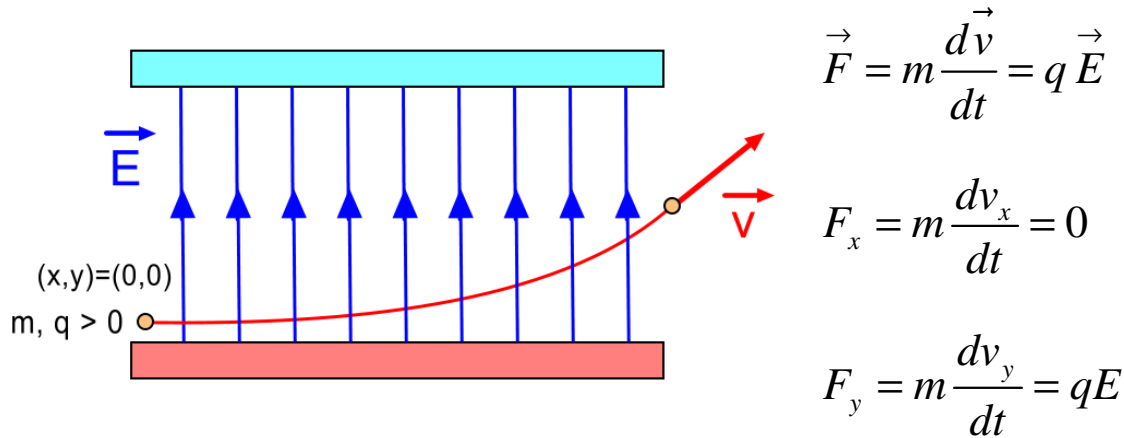
For initial $\vec{v}(0) = v_0\,\hat{i}$, then $v_x = v_0$ and $v_y = \frac{qE}{m}t$.

Integrating again to get x and y results in the two following equations.

$$x(t) = x_0 + v_0 t \qquad\qquad y(t) = y_0 + \frac{qE}{2m}t^2$$

With our initial conditions for the position, these simplify.

$$x(t) = v_0 t \qquad\qquad y(t) = \frac{qE}{2m}t^2$$

PE1 (Practice Problem). Taking the horizontal length of each plate to be l, find the coordinates of the charge as it just leaves the plate region (see figure). Also find the magnitude and direction of the final velocity as the charge leaves the plate region.

2) Constant Magnetic Field (Review from Intro Physics II)

Start with the general form for the Lorentz force: $\vec{F} = \dfrac{d\vec{p}}{dt} = q(\vec{E} + \vec{v} \times \vec{B})$.

Here there is only a magnetic field, which points into the page.
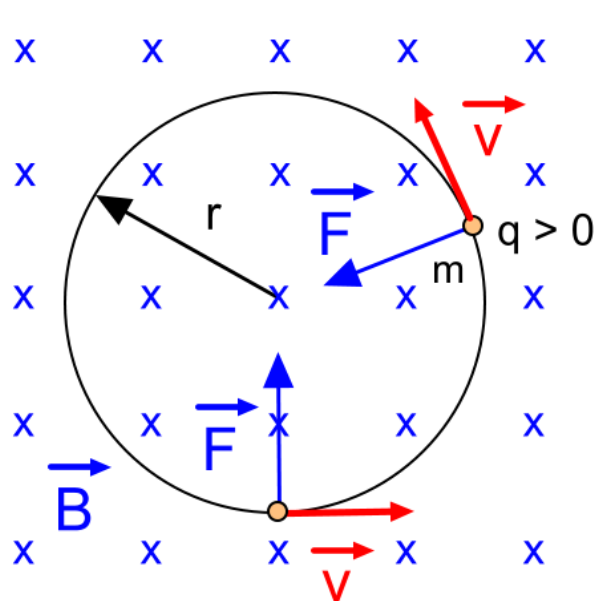
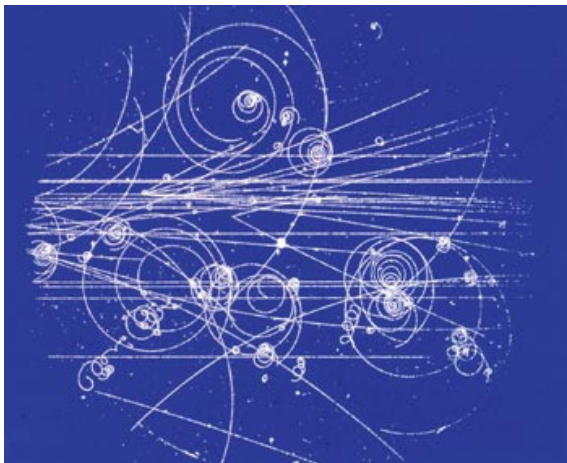$$\vec{F} = m\frac{d\vec{v}}{dt} = q(\vec{v} \times \vec{B})$$

The motion is in a circle.

$$F = m\frac{v^2}{r} = qvB$$

$$m\frac{v}{r} = qB$$

The angular frequency $\omega = \dfrac{v}{r}$ is found to be a constant!

$$\omega = \frac{v}{r} = \frac{qB}{m} .$$

This is called the Larmor frequency. Shooting charged particles perpendicular into a region of constant magnetic field allows one to measure the charge-to-mass ratio q/m from the strength of the magnetic field and the angular frequency.


Courtesy CERN. "A classic bubble chamber photograph. Millions of such interactions were studied during the 1960s and 1970s." CERN

CERN is the European Organization for Nuclear Research, established in 1954. Headquarters is near Geneva.

At the left you see particle tracks where the circular paths indicate motion of charged particles in a magnetic field. Positive charges circle one way and negative the opposite.

E4. Work and Kinetic Energy (Review from Intro Physics I).

The simplest case is applying a constant force along the direction of motion.
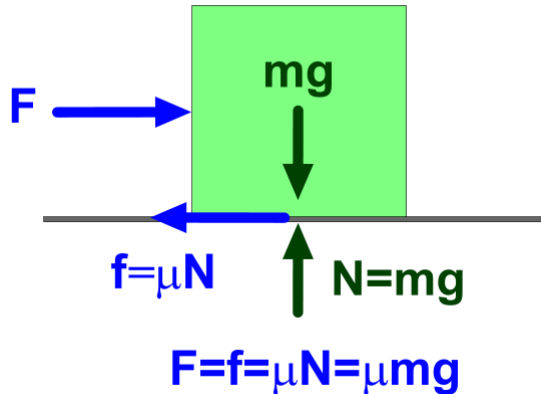
The typical case is one where friction does an equal and opposite amount of work and the speed is constant. In this case, the work done by a person pushing a block with force F and traveling a distance d is

$$W = Fd .$$

For our figure we can substitute to arrive at the following equation.

$$W = Fd = \mu mgd$$

Courtesy Wikimedia. Photo released by the US Navy. Photo taken by Damon J. Moritz.

In the photo, an Office Movers employee pushes boxes out of the office of the Secretary of the Navy as a Yeoman studies for a college course (from Wikimedia).

Work is money. If you were paying someone to work for you, we would not want to give them money for doing no work. Below are two examples of doing no work.

1) Walking around with outstretched arms pushing air like a zombie: F = 0.
$$W = Fd = 0 \cdot d = 0$$

2) Pushing against a wall with a force F but there is no distance traveled: d = 0.
$$W = Fd = F \cdot 0 = 0$$

3) Someone just standing around: F = 0 and d = 0.
$$W = Fd = 0 \cdot 0 = 0$$

In each case the work is zero. So the definition of work in physics makes sense from a practical point of view.

PE2 (Practice Problem). You pay workers to push boxes (crates) containing appliances in a large warehouse to the loading dock. Some crates need more force than others. But some crates are farther from the loading end. What is a reasonable amount of work that a worker can do in one hour? What would be a reasonable amount of pay per N · m or ft · lb? This way you are paying for the actual work done!

PE3 (Practice Problem). Let y = u² and u = 3x. Find y = y(x) and dy/dx. Now calculate dy/dx from the chain rule (dy/du)(du/dx). Do your answers agree?

To allow for non-constant situations we use the calculus in applying a force F(x) for an infinitesimal distance dx and do an integral. We will take the force aligned with x and write

$$W = \int F(x)dx .$$

Lets apply a force to a mass in outer space from rest and then let go of it. What is the work we do?

$$W = \int F\, dx = \int ma\, dx = \int m\frac{dv}{dt}\, dx = \int m\frac{dv}{dx}\frac{dx}{dt}\, dx = \int_0^v mv\, dv = \frac{1}{2}mv^2$$

Do you like our tricks with the chain rule, differentials, and changing the integration variable? This result is called the work-energy theorem, You do work on the mass and this work goes into energy in the form of motion. So we define the energy due to the work we did as the kinetic energy since the mass is now moving through space.

$$E = \frac{1}{2}mv^2 .$$

If you apply a force on an already moving object, the work you do is expressed as a difference of kinetic energies. See below for the more general case of the work-energy theorem.

$$W = \int_{x_1}^{x_2} F\, dx = \int_{v_1}^{v_2} mv\, dv = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2$$

Applying the work definition to the Lorentz force, we use the dot product to project the component of force along the direction of motion.

$$W = \int_1^2 \vec{F}\cdot d\vec{r} = q\int_1^2 (\vec{E} + \vec{v}\times\vec{B})\cdot d\vec{r} = q\int_1^2 \vec{E}\cdot d\vec{r} + q\int_1^2 (\vec{v}\times\vec{B})\cdot d\vec{r} .$$

The last integral is zero, using a chain rule and cycle trick for the triple cross product.

$$q\int_1^2 (\vec{v}\times\vec{B})\cdot \frac{d\vec{r}}{dt}dt = q\int_1^2 (\vec{v}\times\vec{B})\cdot \vec{v}\, dt = \int_1^2 q(\vec{v}\times\vec{v})\cdot \vec{B}\, dt = 0$$

Since the magnetic field affects the direction of the charge's velocity rather than its magnitude, the magnetic field does no work on the charge. Our work-energy theorem from classical mechanics takes this form in electromagnetic theory.

$$W = q\int_1^2 \vec{E}\cdot d\vec{r} = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2$$

E5. Four-Vectors.

The world is four dimensional. You need three spatial coordinates and one time coordinate. So we make a vector with four components where we put time first. But we need to have the same dimension for each coordinate so we use ct instead of t.

$$x^\mu = (ct, x, y, z) \text{ and } dx^\mu = (cdt, dx, dy, dz)$$

Don't worry why we are using superscripts here. The length of the vector in four dimensions is given by taking the squares and inserting a relative minus sign between the time and spatial quadratics. We are using the convention here to put the minus with the spatial coordinates. You could put the minus in front of the time like we did before when we squared ict.

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2 .$$

This is invariant. If we put an object in a primed frame and do not let it move around in that frame, then we have dx' = dy' = dz' = 0. We can then write

$$ds^2 = c^2(dt')^2 - (dx')^2 - (dy')^2 - (dz')^2 = c^2(dt')^2 , \text{ and this must equal}$$

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2 .$$

Therefore, $ds^2 = c^2(dt')^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$. The differential time interval $dt' \equiv d\tau$ is called the proper time since it is the time of the clock in its own frame.

$$ds^2 = c^2 d\tau^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$$

Therefore,

$$c^2 d\tau^2 = c^2 \left[1 - \frac{1}{c^2}\frac{dx^2}{dt^2} - \frac{1}{c^2}\frac{dy^2}{dt^2} - \frac{1}{c^2}\frac{dz^2}{dt^2} \right] dt^2 = c^2 \left[1 - \frac{v^2}{c^2} \right] dt^2$$

A compact result is

$$d\tau = \sqrt{1 - \frac{v^2}{c^2}}\, dt .$$

Do you recognize this? How about in the following forms?

$$dt = \frac{d\tau}{\sqrt{1 - \dfrac{v^2}{c^2}}} \qquad\qquad T = \frac{T_0}{\sqrt{1 - \dfrac{v^2}{c^2}}}$$

It is the differential form of our time-dilation formula. Now we define a special four-velocity u^μ by differentiating our four-vector by the proper time. Here is our four-vector:

$$x^\mu = (ct, x, y, z) .$$

Take the derivative with respect to τ we get

$$u^\mu \equiv \frac{dx^\mu}{d\tau} = (c\frac{dt}{d\tau}, \frac{dx}{d\tau}, \frac{dy}{d\tau}, \frac{dz}{d\tau}) = \frac{1}{\sqrt{1 - \dfrac{v^2}{c^2}}}(c\frac{dt}{dt}, \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt})$$

$$u^\mu = \frac{dx^\mu}{d\tau} = \frac{1}{\sqrt{1 - \dfrac{v^2}{c^2}}}(c, v_x, v_y, v_z)$$

$$u^\mu = \frac{dx^\mu}{d\tau} = \frac{1}{\sqrt{1 - \dfrac{v^2}{c^2}}}(c, \vec{v})$$

It is common practice to define gamma such that $\gamma = \dfrac{1}{\sqrt{1 - \dfrac{v^2}{c^2}}}$.

Then we have $u^\mu = \dfrac{dx^\mu}{d\tau} = \gamma(c, \vec{v})$. You are free to roam in space and have speed $\vec{v}$ in the spatial dimensions but you are force to be pulled forward in the time dimension at the speed of light since $u^0 = c$.

The four-momentum is defined by multiplying the four-velocity by the mass m.

$$p^\mu \equiv m\frac{dx^\mu}{d\tau} = \gamma(mc, \vec{mv})$$

Einstein was led to using the special four-momentum to replace the classical p = mv since he showed that this momentum would be conserved. Also, by taking the derivative with respect to the proper time you have no ambiguity in the time. The proper time is unique, the value the clock reads in the frame it is in. The "length" of each four-vector is invariant from frame to frame.

The four-vectors $x^\mu = (ct, \vec{r})$ and $p^\mu = \gamma(mc, m\vec{v})$. Now define $x_\mu = (ct, -\vec{r})$ and $p_\mu = \gamma(mc, -m\vec{v})$. Then we can give the lengths of the four-vectors as

$$x^\mu x_\mu \equiv c^2 t^2 - r^2 = \tau^2 \text{ (with the Einstein summation convention)},$$

$$p^\mu p_\mu = \gamma^2 \left[m^2 c^2 - m^2 v^2 \right] = \frac{1}{1 - \dfrac{v^2}{c^2}} m^2 c^2 \left[1 - \frac{v^2}{c^2} \right] = m^2 c^2 .$$

The first gives the proper time squared as the invariant and the second one is the mass squared as the invariant with the speed of light squared in there too.

E6. Work and E = mc^2 (Review from Modern Physics).

Start with our four-momentum $p^\mu \equiv m \dfrac{dx^\mu}{d\tau} = \gamma(mc, m\vec{v})$ and from this obtain

$$\vec{p} = m \frac{d\vec{r}}{d\tau} = \gamma m \vec{v} .$$

In one spatial dimension we have

$$p = m \frac{dx}{d\tau} = \gamma m v = \frac{mv}{\sqrt{1 - \dfrac{v^2}{c^2}}} .$$

Then, the work on pushing a mass in outer space (which eliminates friction) is

$$W = \int F\, dx = \int \frac{dp}{dt}\, dx = \int \frac{d}{dt} \left(\frac{mv}{\sqrt{1 - \dfrac{v^2}{c^2}}} \right) dx$$

Use chain-rule and differential tricks similar to those that we did for the classical case to derive the following equivalent form for the work.

$$W = \int \frac{d}{dv}\left(\frac{mv}{\sqrt{1-\dfrac{v^2}{c^2}}}\right) v\, dv$$

Now we will use the integration-by-parts trick. You can remember this trick from the product rule of differentiation given below.

$$\frac{d(fg)}{dv} = g\frac{df}{dv} + f\frac{dg}{dv}$$, which is also $$g\frac{df}{dv} = \frac{d(fg)}{dv} - f\frac{dg}{dv}$$.

So you can move the derivative off the g here and move it to the f.

PE4 (Practice Problem). Use this trick to show the following.

$$W = \int \frac{d}{dv}\left(\frac{mv^2}{\sqrt{1-\dfrac{v^2}{c^2}}}\right) dv - \int \frac{mv}{\sqrt{1-\dfrac{v^2}{c^2}}}\, dv$$

Note that your first integral is no problem as you have a perfect differential. Do the second integral using conventional methods. Do not look this integral up as you can do it easily with conventional methods. Put in your limits of integration which we take from speed 0 to some speed v as we apply the force to the mass initially at rest.

PE5 (Practice Problem). Show that you get

$$W = \left[\frac{mv^2}{\sqrt{1-v^2/c^2}} + mc^2\sqrt{1-v^2/c^2}\right]_0^v.$$

PE6 (Practice Problem). Work out you previous result to arrive at,

$$W = \frac{mv^2}{\sqrt{1-v^2/c^2}} + mc^2\sqrt{1-v^2/c^2} - mc^2$$. Continue the steps to get

$$W = \frac{mc^2}{\sqrt{1-v^2/c^2}} - mc^2$$, which is also $W = E(v) - E(v=0)$.

Therefore, $E = \dfrac{mc^2}{\sqrt{1 - v^2 / c^2}}$. Note immediately that if the speed $v = 0$, your expression

$$E = \frac{mc^2}{\sqrt{1 - v^2 / c^2}} \text{ reduces to } E = mc^2.$$

You still have energy $E = mc^2$ when the mass is at rest. This is called the rest energy. The kinetic energy KE according to Einstein is then the total energy for a moving mass minus the energy it has when it is at rest.

$$KE = \frac{mc^2}{\sqrt{1 - v^2 / c^2}} - mc^2$$

PE7 (Practice Problem). Show that a Taylor Series expansion on the first term in

$KE = \dfrac{mc^2}{\sqrt{1 - v^2 / c^2}} - mc^2$ will lead to the result that at slow speeds Einstein's

relativistic version of the work-energy theorem reduces to Newton's: $E = \dfrac{1}{2} mv^2$.

E7. Energy, Mass, Momentum Triangle (Deriving with Four-Vectors).

You probably saw the "cute triangle" we are going to derive here, but we will do it with our four-vector notation. We saw earlier that the four-momentum is

$$p^\mu = m \frac{dx^\mu}{d\tau} = \gamma(mc, m\vec{v}).$$

Check out $p^0 = \gamma mc$, What's this? Well, you just discovered in the previous section

that $E = \dfrac{mc^2}{\sqrt{1 - v^2 / c^2}}$, which is the same as $E = \gamma mc^2$. So that means

$p^0 = \dfrac{E}{c}$. This is elegant, the fourth dimension here, extending our concept of

momentum is related to the energy. Four dimensions in spacetime means space and time while four dimensions in the momentum arena means energy and momentum.

Summary: $\qquad x^\mu = (ct, \vec{r}), \; u^\mu = \dfrac{dx^\mu}{d\tau} = \gamma(c, \vec{v})$

$$p^\mu = (\frac{E}{c}, \vec{p}), \text{ where } \vec{p} = \frac{m\vec{v}}{\sqrt{1 - \dfrac{v^2}{c^2}}}$$

In the rest frame, $p^\mu = (\dfrac{E}{c}, \vec{0})$. Think of Energy / c as "momentum in time."

Review of our invariant equations. We found earlier that

$$p^\mu p_\mu = m^2 c^2$$

With $p^\mu = (\dfrac{E}{c}, \vec{p})$, we must have

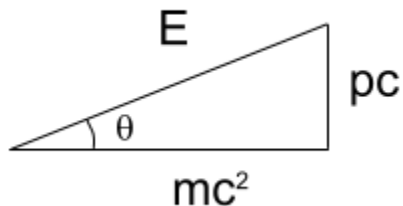$$p^\mu p_\mu = \frac{E^2}{c^2} - p^2 = m^2 c^2,$$

which we can write as

$$E^2 - p^2 c^2 = m^2 c^4 \quad \text{and} \quad E^2 = m^2 c^4 + p^2 c^2.$$

You can remember the formula

$$E^2 = m^2 c^4 + p^2 c^2$$

by a cute right triangle.

At rest, the angle is zero and $E = mc^2$.

At the relativistic extreme the angle is 90° and $E = pc$, which is true for light.