Electromagnetic Theory
Prof. Ruiz, UNC Asheville, doctorphys on YouTube
Chapter F Notes. Faraday's Law

F1. Faraday's Law

In the introductory physics course, second semester, one learns about the four Maxwell equations. If you are a non-physics major or took physics long ago, no problem. We are reviewing all these basic formulas from a novel angle. We derive everything from Coulomb's Law and relativity - well, sort of derive them. Here are the equations we have so far.

$$\overrightarrow{F_E} = \frac{kQq}{r^2}\hat{r} , \quad \overrightarrow{dE} = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r^2}\hat{r}$$ Coulomb's Law and Electric Field

$$\vec{B} = \frac{\mu_0 I}{2\pi r}\hat{\theta}$$ Magnetic field for an infinite line of current.

$$\vec{F} = q(\vec{E} + \vec{v}\times\vec{B})$$ Lorentz Force

We have three additional equations for the field lines and sources of charge and current.

$$\oiint \vec{E}\cdot\overrightarrow{dA} = \frac{Q}{\varepsilon_0}$$ Gauss's Law

$$\oint \vec{B}\cdot\overrightarrow{dl} = \mu_0 I$$ Ampère's Law

$$\oiint \vec{B}\cdot\overrightarrow{dA} = 0$$ No Name

Note that some physics texts give one integral sign for the area integrals rather than the double symbol.

Michael Faraday (1791-1867)
From School of Mathematics and Statistics
University of St. Andrews, Scotland

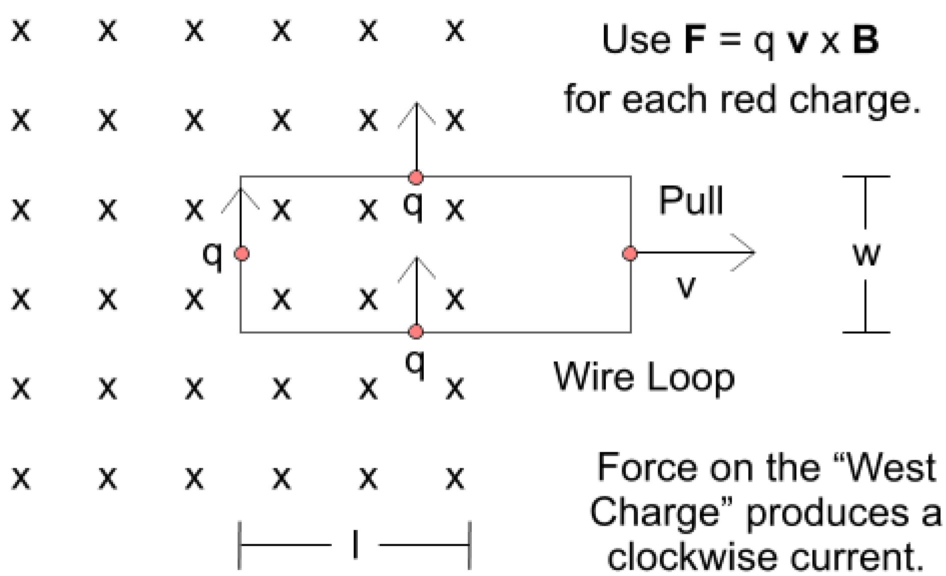
Here is Faraday's Law, which you encountered in intro physics.

$$\oint \vec{E} \cdot \vec{dl} = -\frac{d\Phi_B}{dt},$$

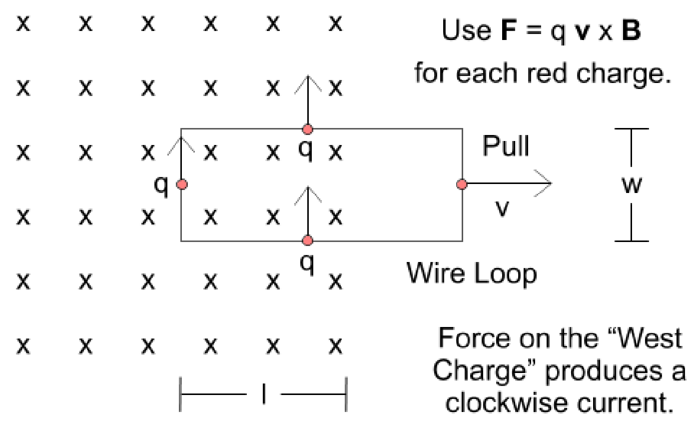
where Φ_B is the magnetic flux. Magnetic flux is found by multiplying the magnetic field with the area through which the field lines penetrate.

$$\Phi_B = BA$$

If the magnetic field is not constant, you have to do an integral. Let's see if we can understand a theoretical argument as to why this is true. We start from what we know. The "x" marks below are the tails of the constant magnetic field B lines that goes into the page. We pull a wire loop through this field.



Apply $\vec{F} = q\vec{v}\, x\, \vec{B}$ to each of the four red positive charges in the wire. These is no force on the east charge since B = 0 there. The other charges are pulled upward but only the west charge starts to move to produce a current due to $F = qvB$. An electric field is generated since we have induced current. The electric field generated must be $E = vB$ since $F = qvB = qE$.



The velocity seen in our formula

$$E = vB$$

is given by $v = -\frac{dl}{dt}$, which is the negative of the decrease in our length portion where there is the magnetic field.

This gives

$$E = vB = -\frac{dl}{dt}B .$$

Now consider the loop integral for this generated E field. The only relevant side is the west side.

$$\oint \vec{E} \cdot \vec{dl} = Ew \text{ (integrating in the direction of the current).}$$

On the west side the electric field lines up with the differential vector length element. We integrate along the path where the electric field is pointing. So we integrate up and therefore get the positive Ew.

The integral for the top part is zero since the E field is perpendicular to the direction which at the top is to the right. There is no E field on the east side. The integral at the bottom is zero similar to the top analysis.

Putting it all together, we obtain

$$\oint \vec{E} \cdot \vec{dl} = -\frac{dl}{dt}Bw .$$

To allow for pulling the wire upwards, we move w into the derivative. The w is constant here but would not be if we pulled upwards instead of the right.

$$\oint \vec{E}\cdot \vec{dl} = -\frac{d(lw)}{dt}B \quad .$$

Note that the product lw is the length times width for where the magnetic field penetrates through the wire. So we call this the area A = lw and write

$$\oint \vec{E}\cdot \vec{dl} = -\frac{d(lw)}{dt}B = -\frac{dA}{dt}B \quad .$$

Since the B is constant we can pull the B into the derivative. But this is significant because if we increased the B field instead of moving the wire, we would get the same effect.

$$\oint \vec{E}\cdot \vec{dl} = -\frac{d(BA)}{dt} \quad .$$

Since $\Phi_B = BA$, the magnetic flux, we have arrived at Faraday's Law from a theoretical analysis.

$$\oint \vec{E}\cdot \vec{dl} = -\frac{d\Phi_B}{dt}$$

Now we have four equations, but we have to add one important piece. We do that in the next chapter. I have chosen the order so that Gauss's Law is first, then the similar surface integral for the magnetic field. Finally you see the line integrals with E first.

$$\oiint \vec{E}\cdot \vec{dA} = \frac{Q}{\varepsilon_0}$$

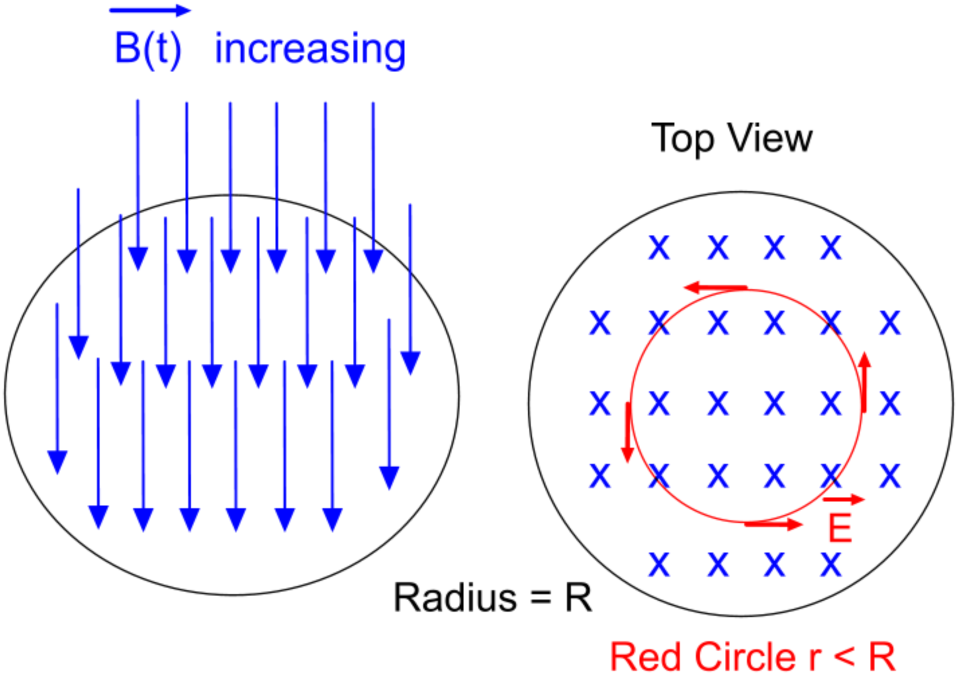
$$\oiint \vec{B}\cdot \vec{dA} = 0$$

$$\oint \vec{E}\cdot \vec{dl} = -\frac{d\Phi_B}{dt}$$

$$\oint \vec{B}\cdot \vec{dl} = \mu_0 I$$

$$\vec{F} = q(\vec{E} + \vec{v}\times\vec{B})$$

F2. Generating an Electric Field in Space



A magnetic field B(t) is increasing through a circular region in space. The B field is downward and from the top view you can see the tails of the vector arrows. There are no wires. Nevertheless, Faraday's Law implies that a circular electric field E will be set up. See the red circle in the right figure. If you had a wire here, then current would flow counterclockwise to oppose the increasing B field. This is Lenz's Law. Nature always opposes you and you get nothing for free. Remember friction - how it always opposes what you are trying to do.

We apply Faraday's Law to the red circle that has the smaller radius r.

$$\oint \vec{E} \cdot \vec{dl} = -\frac{d\Phi_B}{dt} \quad \text{becomes} \quad E(2\pi r) = -\frac{d(BA)}{dt} = -\frac{dB}{dt}\pi r^2$$

The electric field generated by Faraday's Law is $E = -\frac{r}{2}\frac{dB}{dt}$, where the minus sign indicates counterclockwise. For r > R, going outside the magnetic flux region,

$$\oint \vec{E} \cdot \vec{dl} = -\frac{d\Phi_B}{dt} \quad \text{becomes} \quad E(2\pi r) = -\frac{d(BA)}{dt} = -\frac{dB}{dt}\pi R^2 \quad \text{and we get}$$

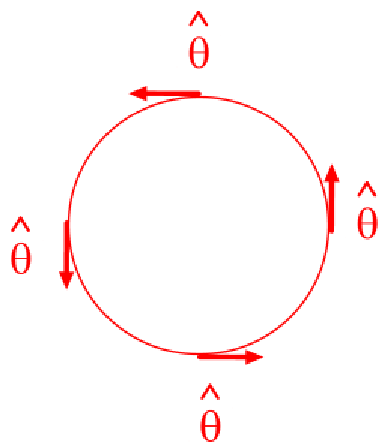
$$E = -\frac{R^2}{2r}\frac{dB}{dt}$$

Summary:

$$r < R \qquad\qquad r > R$$

$$E(r) = -\frac{r}{2}\frac{dB}{dt} \qquad\qquad E(r) = -\frac{R^2}{2r}\frac{dB}{dt}$$

Note that these agree at the boundary $r = R$.



Using the theta unit vector of polar coordinates, we can write the electric field in vector form as follows.

$$\vec{E}(r) = \frac{r}{2}\frac{dB}{dt}\hat{\theta} \quad \text{for} \quad r \le R$$

$$\vec{E}(r) = \frac{R^2}{2r}\frac{dB}{dt}\hat{\theta} \quad \text{for} \quad r \ge R$$

The idea that a changing magnetic field can produce an electric field in space and that the produced electric field is perpendicular to the magnetic field is excellent preparation for our study later of electromagnetic waves.

$$\oint\!\!\oint \vec{E} \cdot \vec{dA} = \frac{Q}{\varepsilon_0}$$

$$\oint\!\!\oint \vec{B} \cdot \vec{dA} = 0$$

$$\oint \vec{E} \cdot \vec{dl} = -\frac{d\Phi_B}{dt}$$

$$\oint \vec{B} \cdot \vec{dl} = \mu_0 I$$

These two main additional points here are:

1) a changing magnetic field creates an electric field,

2) the created electric field is perpendicular to the direction of the magnetic field.

The missing piece to the puzzle in the equations at the left is the law that states a changing electric field creates a magnetic field - the reverse of the problem we just analyzed..

This piece Maxwell added, as we will see in our next chapter. Having the symmetry that changing B creates E and changing E creates B allows for the self propagation of E and B fields in the form of waves - electromagnetic (EM) waves. Consider this a preview of wonderful things to come.

F3. Lenz's Law

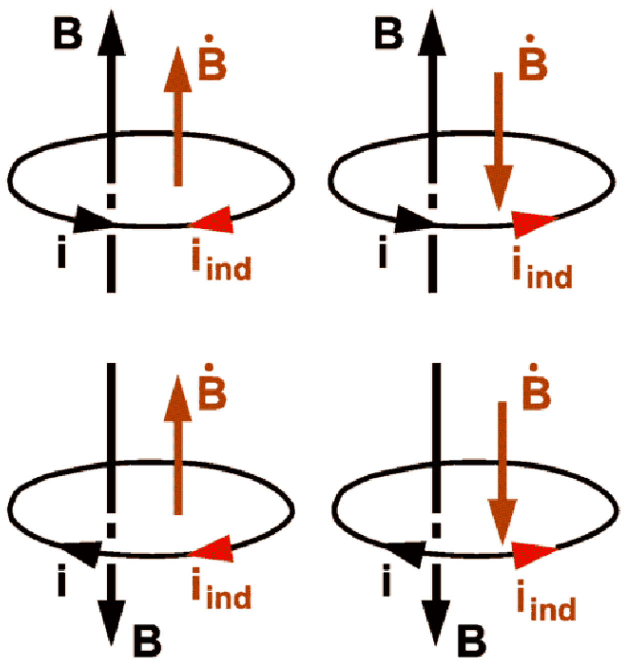


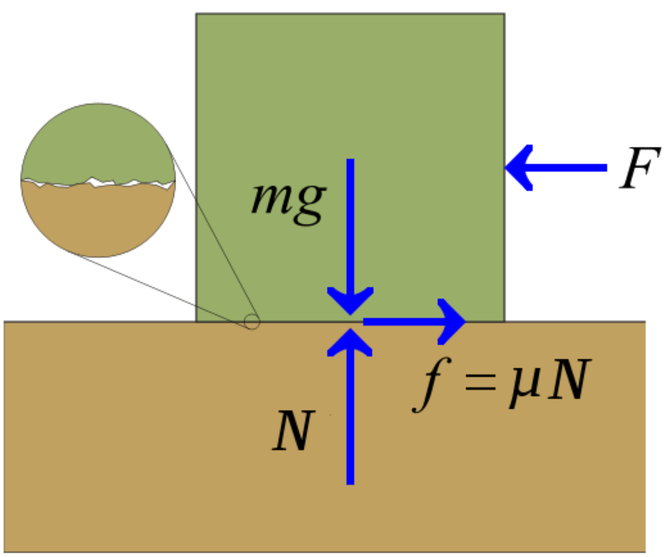
Figure Courtesy P.wormer, Wikimedia Commons.

$$\oint \vec{E} \cdot \vec{dl} = -\frac{d\Phi_B}{dt}$$

The minus sign you see in Faraday's Law is called Lenz's Law. This is illustrated at the left with a current loop (black) that sets up a an original magnetic field (black B).

If you increase the original current, the change in B (B dot) is in the same direction of your original B, but the induced current (red) will oppose the direction of your original current.

The induced current sets up its own magnetic field that opposes what your original magnetic field is trying to do. If you try to increase your original B, then the induced B subtracts from your original magnetic field. If you try to decrease your original B, then your induced B adds to your original magnetic field.



Friction Image Courtesy Keta (Pieter Kuiper), Wikimedia

Nature always opposes what you are trying to do. This is like friction. Friction always works against you.

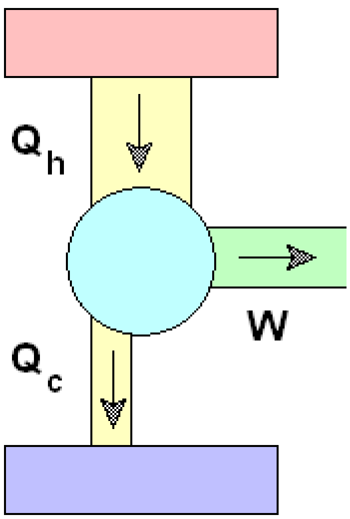
For cases of no acceleration, we have the usual equilibrium equations.

$$N = mg$$

$$F = f = \mu N = \mu mg$$

The First Law of Thermodynamics

The Second Law of Thermodynamics



According to the second law of thermodynamics you cannot use 100% of the available heat energy from the hot source Q_h. So you have to dump some, Qc into the cooler environment. The useful work W you get to drive your car is given by

$$W = Q_h - Q_c \, .$$

Your wasted energy is your thermal pollution.

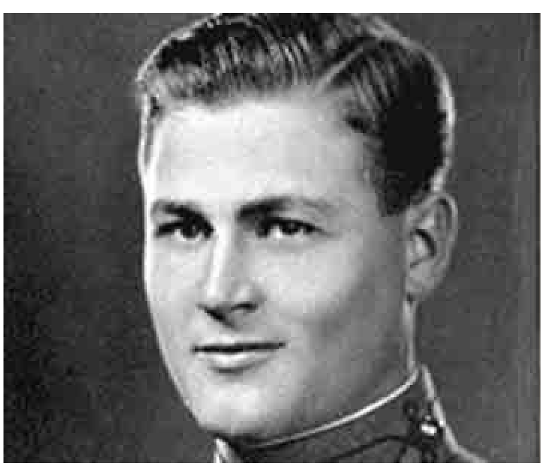


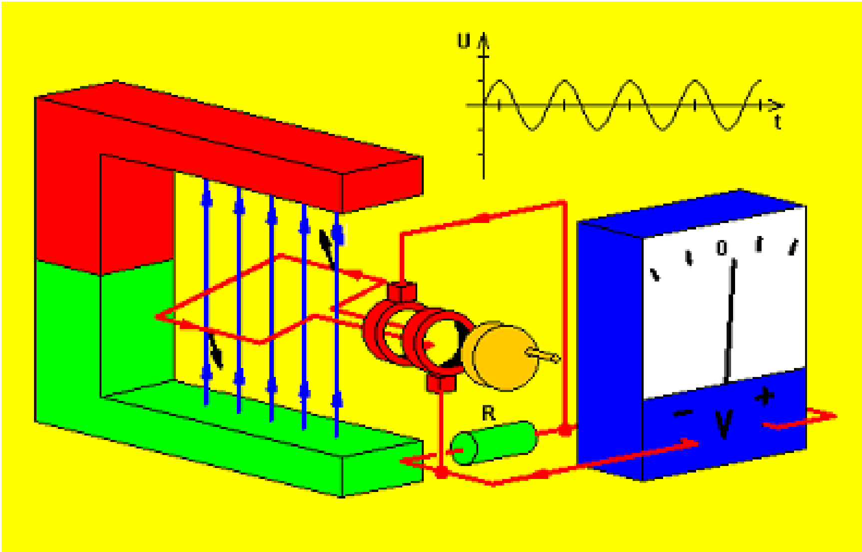
Photo from www.johnhclouse.com

Edward Aloysius Murphy, Jr. (1918-1990)

Murphy's Law:

If anything can go wrong, it will.

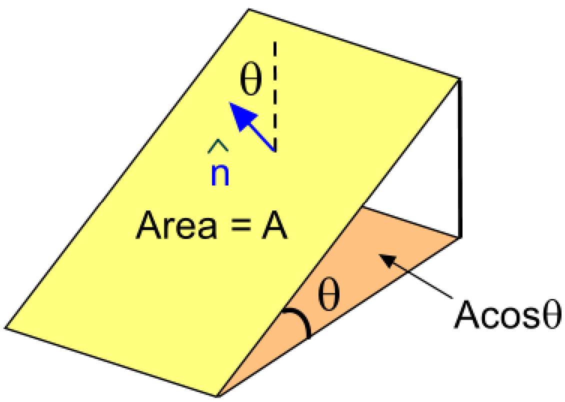
F4. The Generator



Courtesy Walter Fendt

Turn the crank evenly and you get a sine wave. When the magnetic flux decreases through the loop as seen in the instant at the left, current is generated to produce a magnetic field in the same direction as the magnet's field. Current at this instant is then clockwise looking down from the top.

$$\mathcal{E} \equiv \oint \vec{E} \cdot \vec{dl} = -\frac{d\Phi_B}{dt} \quad \text{where} \quad \Phi_B = \iint \vec{B} \cdot \vec{dA}$$



From the above figure we see that $\vec{B} = B\hat{k}$. Then

$$\Phi_B = \iint B\hat{k} \cdot \hat{n}\, dA$$

$$\Phi_B = B\cos\theta \iint dA = BA\cos\theta$$

$$\mathcal{E} = \oint \vec{E} \cdot \vec{dl} = -\frac{d(BA\cos\theta)}{dt} = -BA\frac{d\cos\theta}{dt}.$$ Take $\theta = \omega t$.

$$\mathcal{E} = -BA\frac{d\cos\omega t}{dt} = -BA(-\omega\sin\omega t) = BA\sin\omega t$$

If you have N loops of wire, $\mathcal{E} = \oint \vec{E} \cdot \vec{dl} = -N\frac{d\Phi_B}{dt} = NBA\sin\omega t$.