Electromagnetic Theory
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Chapter I Notes. The Maxwell Equations in Differential Form

I1. The Maxwell Equations in Differential Form

We will now transform the integral form of the Maxwell equations into differential form.

$$\oint\!\!\oint \vec{E} \cdot \overrightarrow{dA} = \frac{Q}{\varepsilon_0}$$

$$\oint\!\!\oint \vec{B} \cdot \overrightarrow{dA} = 0$$

$$\oint \vec{B} \cdot \overrightarrow{dl} = \mu_0 I + \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt}$$

$$\oint \vec{E} \cdot \overrightarrow{dl} = -\frac{d\Phi_B}{dt}$$

1. The First Maxwell Equation

$$\oint\!\!\oint \vec{E} \cdot \overrightarrow{dA} = \frac{Q}{\varepsilon_0}$$

Express the left side using the Divergence Theorem.

$$\oint\!\!\oint \vec{E} \cdot \overrightarrow{dA} = \iiint_V \nabla \cdot \vec{E}\, dV$$

Express the right side with the volume charge density.

$$\frac{Q}{\varepsilon_0} = \iiint_V \frac{\rho}{\varepsilon_0} dV \qquad \text{Then, } \iiint_V \nabla \cdot \vec{E}\, dV = \iiint_V \frac{\rho}{\varepsilon_0} dV .$$

A rigorous analysis requires us to write it this way:

$$\iiint_V (\nabla \cdot \vec{E} - \frac{\rho}{\varepsilon_0})\, dV = 0 .$$

Then we state that since the volume integration is arbitrary, i.e., we can take different volumes, the integrand must vanish to make the equation true in general.

Arbitrary volumes mean that the following

$$\iiint_V (\nabla \cdot \vec{E} - \frac{\rho}{\varepsilon_0})\, dV = 0$$

implies

$$\nabla \cdot \vec{E} - \frac{\rho}{\varepsilon_0} = 0 ,$$

which leads to

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0} .$$

This the differential form for Gauss's Law, which in turn is equivalent to Coulomb's Law.

2. The Second Maxwell Equation

This one is easy after doing the first. Since

$$\oiint \vec{E} \cdot \overrightarrow{dA} = \frac{Q}{\varepsilon_0} \text{ becomes } \nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0} ,$$

$$\oiint \vec{B} \cdot \overrightarrow{dA} = 0 \text{ becomes } \nabla \cdot \vec{B} = 0 .$$

No magnetic field lines can originate at a point since there are no magnetic monopoles. Therefore, there are no diverging magnetic field lines from a point. This is a most elegant statement that there are no magnetic monopoles. The magnetic field tends to loop and the presence of a north and south pole for a magnet means we have a cancellation effect. In other words, there is no such thing as magnetic charge, at least so far as we know.

If we ever find a magnetic monopole, then this basic equation will have to be modified. And if so, which other Maxwell equation needs to be modified to account for a current of moving monopoles? Your answer can be checked by perfect symmetry in the Maxwell equations: charge, electrical current, monopoles (magnetic charge), and magnetic currents

3. The Third Maxwell Equation

What about this one?

$$\oint \vec{B} \cdot \vec{dl} = \mu_0 I + \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt}$$

We use Stoke's theorem for the left side.

$$\oint \vec{B} \cdot \vec{dl} = \iint_A (\nabla \times \vec{B}) \cdot \vec{dA}$$

Then we need to express the right side $\mu_0 I + \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt}$ as an area integral. We use the definition of the current density. If you are hazy on this from your intro physics course, we are led to it here. The mathematics guides us and suggests the following definition:

$$I = JA \quad \text{and in general} \quad I = \iint_A \vec{J} \cdot \vec{dA}.$$

The flux Φ_E is no problem because an area is involved in its definition already:

$$\Phi_E = EA \quad \text{and in general} \quad \Phi_E = \iint_A \vec{E} \cdot \vec{dA}.$$

Putting this all together:

$$\iint_A (\nabla \times \vec{B}) \cdot \vec{dA} = \mu_0 \iint_A \vec{J} \cdot \vec{dA} + \mu_0 \varepsilon_0 \frac{d}{dt} \iint_A \vec{E} \cdot \vec{dA}.$$

We move the derivative inside the integral since the integration is over area and has nothing to do with time. We write as a partial derivative as E depends on x, y, z, and t.

$$\iint_A (\nabla \times \vec{B}) \cdot \vec{dA} = \mu_0 \iint_A \vec{J} \cdot \vec{dA} + \mu_0 \varepsilon_0 \iint_A \frac{\partial \vec{E}}{\partial t} \cdot \vec{dA}$$

We now rewrite

$$\iint_A (\nabla \times \vec{B}) \cdot \overrightarrow{dA} = \mu_0 \iint_A \vec{J} \cdot \overrightarrow{dA} + \mu_0 \varepsilon_0 \iint_A \frac{\partial \vec{E}}{\partial t} \cdot \overrightarrow{dA}$$

as

$$\iint_A \left[(\nabla \times \vec{B}) - \mu_0 \vec{J} - \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \right] \cdot \overrightarrow{dA} = 0$$

Since the surface area chosen is arbitrary, the integrand must vanish to make this true in general. This gives us the third Maxwell equation.

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

4. The Fourth Maxwell Equation

The last Maxwell Equation is easy since it is similar and simpler than the third. Since

$$\oint \vec{B} \cdot \overrightarrow{dl} = \mu_0 I + \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt} \quad \text{becomes} \quad \nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \,,$$

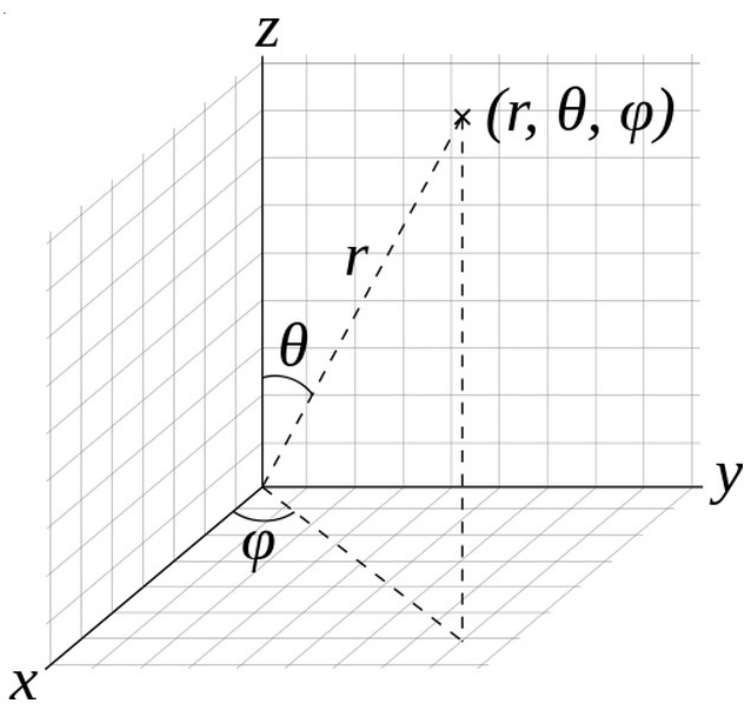
$$\oint \vec{E} \cdot \overrightarrow{dl} = -\frac{d\Phi_B}{dt} \quad \text{becomes} \quad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \,.$$

The Maxwell Equations in Integral Form (left) and Differential Form (right)

$\oiint \vec{E}\cdot d\vec{A} = \frac{Q}{\varepsilon_0}$	$\nabla\cdot\vec{E} = \frac{\rho}{\varepsilon_0}$
$\oiint \vec{B}\cdot d\vec{A} = 0$	$\nabla\cdot\vec{B} = 0$
$\oint \vec{B}\cdot d\vec{l} = \mu_0 I + \mu_0\varepsilon_0 \frac{d\Phi_E}{dt}$	$\nabla\times\vec{B} = \mu_0\vec{J} + \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}$
$\oint \vec{E}\cdot d\vec{l} = -\frac{d\Phi_B}{dt}$	$\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$

I2. Insight into the Divergence

Let's see if we can gain some insight into the divergence by investigating $\nabla\cdot\vec{E} = \frac{\rho}{\varepsilon_0}$ where we have a point charge. Therefore, $\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^2}\hat{r}$. We want to do the calculation in Cartesian coordinates, so we express $\hat{r}$ in terms of $\hat{i}$, $\hat{j}$, and $\hat{k}$.



Courtesy Andeggs, Wikimedia.

From the left figure you see we can form the vector along r by setting

$$\vec{r} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k}\,.$$

The unit vector along the radial direction is then

$$\hat{r} = \frac{x}{r}\hat{i} + \frac{y}{r}\hat{j} + \frac{z}{r}\hat{k}$$

Note that $\hat{r}\cdot\hat{r} = 1$. Why?

The electric field $\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^2}\hat{r}$ with $\hat{r} = \frac{x}{r}\hat{i} + \frac{y}{r}\hat{j} + \frac{z}{r}\hat{k}$ becomes.

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^2}\left[\frac{x}{r}\hat{i} + \frac{y}{r}\hat{j} + \frac{z}{r}\hat{k}\right] = \frac{1}{4\pi\varepsilon_0}\left[\frac{x}{r^3}\hat{i} + \frac{y}{r^3}\hat{j} + \frac{z}{r^3}\hat{k}\right].$$

We could use $r^2 = x^2 + y^2 + z^2$ now, but it is best not to do this in order to keep our notation concise. Now we are ready to take the divergence.

$$\nabla \cdot \vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = \frac{1}{4\pi\varepsilon_0}\left[\frac{\partial}{\partial x}(\frac{x}{r^3}) + \frac{\partial}{\partial y}(\frac{y}{r^3}) + \frac{\partial}{\partial z}(\frac{z}{r^3})\right]$$

The three derivatives are similar so work with the first one.

$$\frac{\partial}{\partial x}(\frac{x}{r^3}) = \frac{1}{r^3} + x\frac{\partial}{\partial x}\frac{1}{r^3} = \frac{1}{r^3} + x\frac{\partial}{\partial r}(\frac{1}{r^3})\frac{\partial r}{\partial x}$$

The two parts of the second term are $x\frac{\partial}{\partial r}(\frac{1}{r^3}) = -\frac{3x}{r^4}$ and

$$\frac{\partial r}{\partial x} = \frac{\partial}{\partial x}\sqrt{x^2 + y^2 + z^2} = \frac{1}{2}\frac{1}{\sqrt{x^2 + y^2 + z^2}}(2x) = \frac{x}{r}.$$

PI1 (Practice Problem). Show this quickly by implicit differentiation of r^2.

Putting it all together, $\frac{\partial}{\partial x}(\frac{x}{r^3}) = \frac{1}{r^3} - \frac{3x^2}{r^5}$. Finally we get the divergence below.

$$\nabla \cdot \vec{E} = \frac{1}{4\pi\varepsilon_0}\left[3\frac{1}{r^3} - \frac{3x^2}{r^5} - \frac{3y^2}{r^5} - \frac{3z^2}{r^5}\right] = \frac{1}{4\pi\varepsilon_0}\left[\frac{3}{r^3} - \frac{3}{r^3}\right] = 0$$

But Wait! Why didn't we get $\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$ as there is charge somewhere?

You will see why. Read on.

Let's try inside a uniform sphere of charge. From before we know

$$\oiint \vec{E}\cdot d\vec{A} = \frac{Q}{\varepsilon_0}, \quad E(4\pi r^2) = \frac{1}{\varepsilon_0}\rho\frac{4}{3}\pi r^3, \text{ and } \vec{E} = \frac{\rho}{3\varepsilon_0} r\,\hat{r} .$$

Then,

$$\vec{E} = \frac{\rho}{3\varepsilon_0} r\left[\frac{x}{r}\hat{i} + \frac{y}{r}\hat{j} + \frac{z}{r}\hat{k}\right] = \frac{\rho}{3\varepsilon_0}\left[x\,\hat{i} + y\,\hat{j} + z\,\hat{k}\right].$$

$$\nabla\cdot\vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = \frac{\rho}{3\varepsilon_0}\left[\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}\right] = \frac{\rho}{3\varepsilon_0}(3) = \frac{\rho}{\varepsilon_0}$$

Now we get $\nabla\cdot\vec{E} = \dfrac{\rho}{\varepsilon_0}$ and that's because charge density is actually at our location.

This is a profound point! We now correctly understand the first Maxwell equation!

Outside the charge in space, away from the charge, you get $\nabla\cdot\vec{E} = 0$ even though you have an electric field out there. This is a deep discovery into the meaning of the differential form for Gauss's Law. In empty space, you get zero, but when you are in the charge-density region you get the nonzero value.

I3. Insight into the Curl

Let's see if we can gain some insight into the curl by investigating

$\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$ where $\frac{\partial \vec{E}}{\partial t} = 0$. We only have current in a wire.

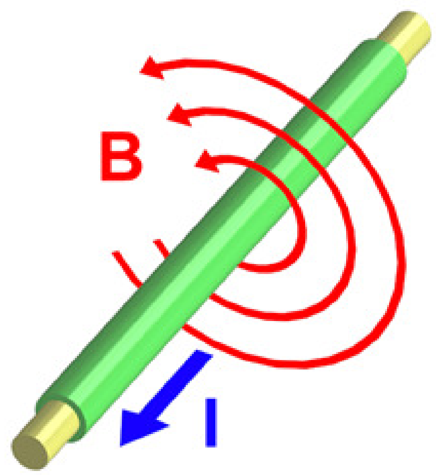
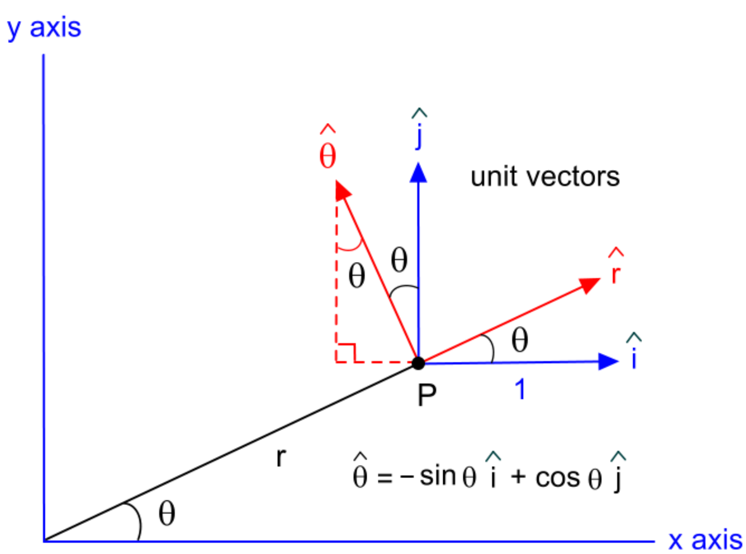


Image Credit: Wikimedia, from User: Stannered from an original by User: Wapcaplet.

Recall our magnetic field produced by a current in a wire.

$$\vec{B} = \frac{\mu_0 I}{2\pi r} \hat{\theta}$$

We will calculate the curl in $\nabla \times \vec{B} = \mu_0 \vec{J}$ calculation in Cartesian coordinates. This means we need to express $\hat{\theta}$ in terms of $\hat{i}$ and $\hat{j}$.



You find the usual Cartesian unit vectors $\hat{i}$ and $\hat{j}$ in the left figure as well as the polar unit vectors $\hat{r}$ and $\hat{\theta}$,

All these unit vectors point in increasing directions of their respective coordinates.

From the right triangle in red, we arrive at the expression of $\hat{\theta}$ in terms of $\hat{i}$ and $\hat{j}$:

$\hat{\theta} = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}$, which also has the required unit length. Why?

While we are here, we can obtain the result $\hat{r} = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$. Our two equations are below.

$$\hat{r} = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$$

$$\hat{\theta} = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}$$

Note that $\hat{r}\cdot\hat{r} = \hat{\theta}\cdot\hat{\theta} = 1$ and $\hat{r}\cdot\hat{\theta} = 0$. Below are your regular transformations between polar and Cartesian coordinates which you encountered in math at some point before.

$$x = r\cos\theta \qquad y = r\sin\theta$$

$$r = \sqrt{x^2 + y^2} \qquad \theta = \tan^{-1}\frac{y}{x}$$

PI2 (Practice Problem). Find $\hat{i}$ and $\hat{j}$ in terms of $\hat{r}$ and $\hat{\theta}$.

Expressing $\vec{B} = \dfrac{\mu_0 I}{2\pi r}\hat{\theta}$ with $\hat{\theta} = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}$, we obtain

$$\vec{B} = \frac{\mu_0 I}{2\pi r}\left[-\sin\theta\,\hat{i} + \cos\theta\,\hat{j}\right].$$

Now use $x = r\cos\theta$ and $y = r\sin\theta$, i.e., $\cos\theta = \dfrac{x}{r}$ and $\sin\theta = \dfrac{y}{r}$.

$$\vec{B} = \frac{\mu_0 I}{2\pi r}\left[-\frac{y}{r}\hat{i} + \frac{x}{r}\hat{j}\right]$$

$$\vec{B} = \frac{\mu_0 I}{2\pi}\left[-\frac{y}{r^2}\hat{i} + \frac{x}{r^2}\hat{j}\right]$$

Take the curl. Note that r in this case is the polar coordinate and not the spherical coordinate we encountered in the Gaussian analysis.

$$\nabla \times \vec{B} = \frac{\mu_0 I}{2\pi} \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\frac{y}{r^2} & \frac{x}{r^2} & 0 \end{vmatrix} = \frac{\mu_0 I}{2\pi} \hat{k} \left[\frac{\partial}{\partial x}(\frac{x}{r^2}) + \frac{\partial}{\partial y}(\frac{y}{r^2}) \right]$$

PI3 (Practice Problem). Why is there only a z-component for this curl?

First consider $\frac{\partial}{\partial x}(\frac{x}{r^2}) = \frac{1}{r^2} + x\frac{\partial}{\partial x}(\frac{1}{r^2}) = \frac{1}{r^2} + x(\frac{-2}{r^3})\frac{\partial r}{\partial x}$

The last derivative is $\frac{\partial r}{\partial x} = \frac{x}{r}$, using $r^2 = x^2 + y^2$ and $2rdr = 2xdx + 2ydy$.

This last step is the implicit-differentiation trick in two dimensions x and y.

Then $\frac{\partial}{\partial x}(\frac{x}{r^2}) = \frac{1}{r^2} + x(\frac{-2}{r^3})(\frac{x}{r}) = \frac{1}{r^2} - 2\frac{x^2}{r^4}$ and

$$\nabla \times \vec{B} = \frac{\mu_0 I}{2\pi} \hat{k} \left[\frac{1}{r^2} - 2\frac{x^2}{r^4} + \frac{1}{r^2} - 2\frac{y^2}{r^4} \right]$$

$$\nabla \times \vec{B} = \frac{\mu_0 I}{2\pi} \hat{k} \left[\frac{2}{r^2} - 2\frac{(x^2 + y^2)}{r^4} \right] = \frac{\mu_0 I}{2\pi} \hat{k} \left[\frac{2}{r^2} - \frac{2r^2}{r^4} \right] = 0$$

But Wait! Why didn't we get $\nabla \times \vec{B} = \mu_0 \vec{J}$ as there is current somewhere?

You will see why. Read on.

Let's try inside a uniform wire of current. From before we know

$\oint \vec{B} \cdot \vec{dl} = \mu_0 I$, $B(2\pi r) = \mu_0 J \pi r^2$, and $\vec{B} = \frac{\mu_0 J}{2} r \hat{\theta}$.

Using $\hat{\theta} = -\sin\theta \, \hat{i} + \cos\theta \, \hat{j}$, we have

$$\vec{B} = \frac{\mu_0 J}{2} r \hat{\theta} = \frac{\mu_0 J}{2} r \left[-\sin\theta \, \hat{i} + \cos\theta \, \hat{j} \right]$$

$$\vec{B} = \frac{\mu_0 J}{2} \left[-r\sin\theta \, \hat{i} + r\cos\theta \, \hat{j} \right] = \frac{\mu_0 J}{2} \left[-y \, \hat{i} + x \, \hat{j} \right]$$

Then,

$$\nabla \times \vec{B} = \frac{\mu_0 J}{2} \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & 0 \end{vmatrix} = \frac{\mu_0 J}{2} \hat{k} \left[\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} \right] = \frac{\mu_0 J}{2} \hat{k} \cdot 2$$

$$\nabla \times \vec{B} = \mu_0 J \, \hat{k}$$

$$\nabla \times \vec{B} = \mu_0 \vec{J}$$

We now get the nonzero current density since we are at a point where current density actually exists.

We now correctly understand the Maxwell equation wit current sources!