Electromagnetic Theory
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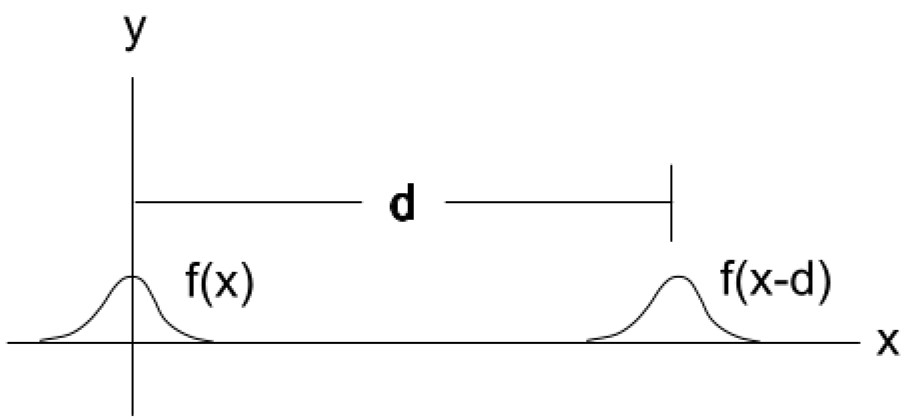
Chapter J Notes. The Wave Equation

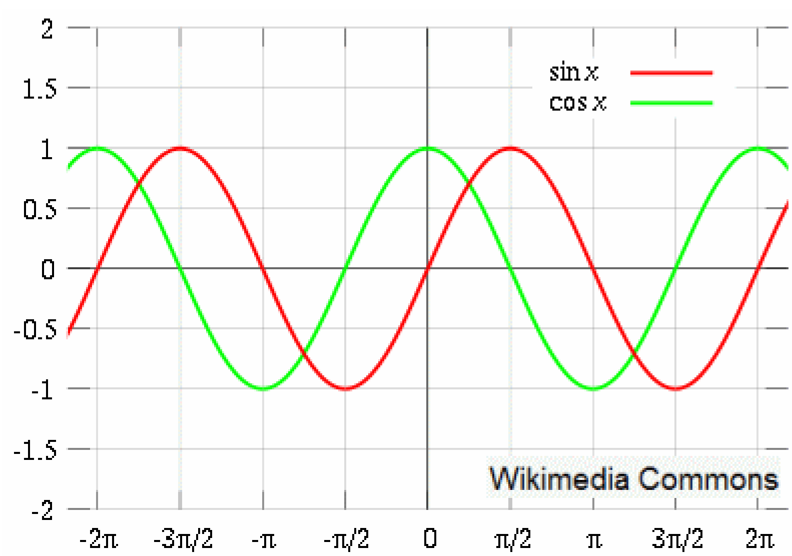
J1. The Wave Equation

A function $y = f(x)$ is shown with a peak at $f(0)$. Denote this by writing $f(0) = peak$. If we shift this function to the right by a distance d, then the new function $h(x)$ must be $h(x) = f(x-d)$. Here is how you can check this rule. Is the peak now at $x = d$? Does $h(d) = peak$? We do this check below the figure.



$$f(0) = peak \quad \text{and} \quad h(x) = f(x-d)$$

$$h(d) = f(d-d) = f(0) = peak$$



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It checks out. Do you remember doing the shift trick in trigonometry? If you shift the cosine by $\pi/2$ to the right, you get the sine.

$$\sin x = \cos(x - \frac{\pi}{2})$$

The above relation also tells you that the sine of an angle in a right triangle equals the cosine of its complement.

Since $f(x-d)$ is our shifted function to the right by a distance d, we can let $d = vt$ to obtain a traveling function to the right. Let's search for a differential equation for this function, i.e., we want a differential equation such that our traveling wave $f(x-vt)$ is the solution. Common practice is to use ψ for a wave. So we write

$\psi(x,t) = f(x-vt)$, defining $u = x - vt$. Note that $\dfrac{\partial u}{\partial x} = 1$ and $\dfrac{\partial u}{\partial t} = -v$.

Then we take derivatives in our quest for the "magic" differential wave equation,

$$\frac{\partial \psi(x,t)}{\partial x} = \frac{\partial f(x-vt)}{\partial x} = \frac{\partial f(u)}{\partial x} = \frac{df(u)}{du}\frac{\partial u}{\partial x} = \frac{df(u)}{du}\cdot 1 = \frac{df(u)}{du}$$

$$\frac{\partial \psi(x,t)}{\partial t} = \frac{\partial f(x-vt)}{\partial t} = \frac{\partial f(u)}{\partial t} = \frac{df(u)}{du}\frac{\partial u}{\partial t} = \frac{df(u)}{du}\cdot(-v).$$

We can now put together the following differential equation from the above. We find

$$\frac{\partial \psi(x,t)}{\partial x} = -\frac{1}{v}\frac{\partial \psi(x,t)}{\partial t} \text{ and write } \frac{\partial \psi_R(x,t)}{\partial x} = -\frac{1}{v}\frac{\partial \psi_R(x,t)}{\partial t},$$

adding the subscript R for "Right" to emphasize that this wave is traveling down the x axis in the positive direction.

But for the wave traveling to the left, we must have the same equation with the velocity in the negative direction. This reverses the sign in front of v since u in that case would be u = x + vt with f(u) = f(x+vt).

$$\frac{\partial \psi_L(x,t)}{\partial x} = +\frac{1}{v}\frac{\partial \psi_L(x,t)}{\partial t}.$$

This is not acceptable because now we have two differential equations. There should be nothing special about right or left. We want a differential equation where the sign does not matter. So we proceed to the second derivative.

We start with

$$\psi(x,t) = f(x - vt) \quad \text{and} \quad u = x - vt \,.$$

We already found $\dfrac{\partial \psi(x,t)}{\partial x} = \dfrac{df(u)}{du}$ and $\dfrac{\partial \psi(x,t)}{\partial t} = -v\dfrac{df(u)}{du}$.

Now we take the second derivatives with respect to x and t.

$$\frac{\partial^2 \psi(x,t)}{\partial x^2} = \frac{\partial}{\partial x}\frac{df(u)}{du} = \frac{d^2 f(u)}{du^2}\frac{\partial u}{\partial x} = \frac{d^2 f(u)}{du^2}$$

$$\frac{\partial^2 \psi(x,t)}{\partial t^2} = \frac{\partial}{\partial t}\left[-v\frac{df(u)}{du}\right] = -v\frac{d^2 f(u)}{du^2}\frac{\partial u}{\partial t} = v^2\frac{d^2 f(u)}{du^2} \,.$$

This leads to these.

$$\boxed{\frac{\partial^2 \psi(x,t)}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 \psi(x,t)}{\partial t^2}} \qquad \boxed{\frac{\partial^2 \psi}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 \psi}{\partial t^2}}$$

Note that when you square plus or minus v you get positive v squared. This differential equation applies to waves moving to the left or to the right. This is the wave equation in one dimension. The general solution is a combination of a wave moving right and one moving left.

$$\psi(x,t) = Af(x - vt) + Bg(x + vt)$$

Note that a second-order differential equation has two solutions. Here, we have one wave traveling to the left and one to the right. For the wave equation in three dimensions where $\psi = \psi(x, y, z, t)$, we have

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 \psi}{\partial t^2} \,.$$

With the del operator ∇, we can write this very elegantly. First note that since

$$\nabla \equiv \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k},$$

we can write

$$\nabla \cdot \nabla = \left[\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right] \cdot \left[\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right]$$

$$\nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

We see now that it makes sense to have the shorthand definition we made earlier.

$$\nabla^2 \equiv \nabla \cdot \nabla$$

The symbol ∇^2 is also called the Laplacian operator, which we have encountered before. So our

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 \psi}{\partial t^2}$$

can be neatly written as follows.

$$\boxed{\nabla^2 \psi = \frac{1}{v^2}\frac{\partial^2 \psi}{\partial t^2}}$$

You can remember where the v goes from dimensional analysis. Since distance equals velocity times time, your velocity has to go with the time t. Since we have the second derivative, think of distance as being squared and time as being squared. So you need the velocity squared.

J2. Light Waves - A Basic Derivation

We give you here a basic derivation of light waves from the Maxwell equations in integral form. In the next class we will provide a sophisticated derivation in three spatial dimensions using vector calculus.

Our desire is to see what happens in free space. Now free space does not mean no electric and magnetic fields. Free space instead simply means a region of space where there are no charges and currents. So we set $Q = 0$ and $I = 0$ in the Maxwell equations. The resulting free space equations are at the right.

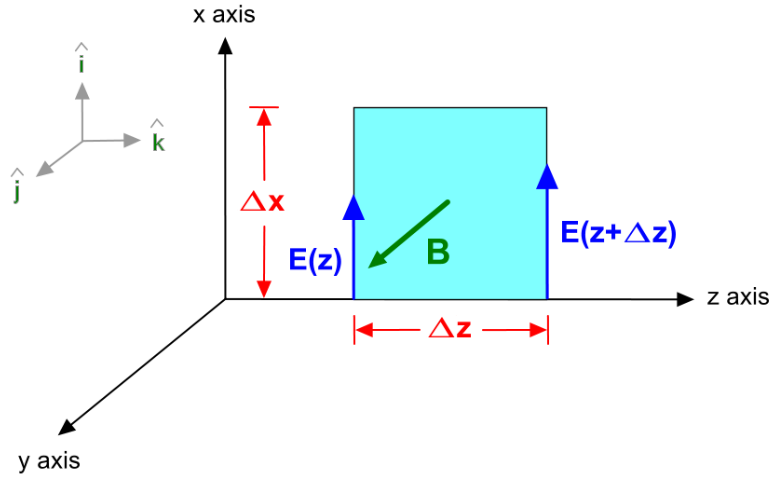
Maxwell Equations	Maxwell Equations in Free Space
$\oiint \vec{E} \cdot \overrightarrow{dA} = \frac{Q}{\varepsilon_0}$	$\oiint \vec{E} \cdot \overrightarrow{dA} = 0$
$\oiint \vec{B} \cdot \overrightarrow{dA} = 0$	$\oiint \vec{B} \cdot \overrightarrow{dA} = 0$
$\oint \vec{E} \cdot \overrightarrow{dl} = -\frac{d\Phi_B}{dt}$	$\oint \vec{E} \cdot \overrightarrow{dl} = -\frac{d\Phi_B}{dt}$
$\oint \vec{B} \cdot \overrightarrow{dl} = \mu_0 I + \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt}$	$\oint \vec{B} \cdot \overrightarrow{dl} = \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt}$

The two free-space Maxwell equations we will be using are the last two - the two with the line integrals - the two equations that are nonzero.

These are Faraday's Law and the last piece of the puzzle that Maxwell added. The Maxwell addition gives us a similar equation to Faraday's Law with B and E interchanged, an additional constant, and an opposite sign. There is no "Lenz's Law" type minus sign in the Maxwell piece.

$$\oint \vec{E} \cdot \overrightarrow{dl} = -\frac{d\Phi_B}{dt} \qquad\qquad \oint \vec{B} \cdot \overrightarrow{dl} = \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt}$$

We work with these next.



We start with an electric field that lies along the z axis but pointing along the x axis.

We calculate

$$\oint \vec{E} \cdot \vec{dl} = -\frac{d\Phi_B}{dt}$$

where $\Phi_B = \iint \vec{B} \cdot \vec{dA}$.

We take B pointing along the normal to dA and if it turns out to be negative, a minus sign will appear. We want to do the loop integral consistent with B pointing out. So we want to go counterclockwise using the right-hand rule. The only contributions are where the electric fields are parallel to the line segments.

$$\oint \vec{E} \cdot \vec{dl} = E(z+\Delta z)\Delta x - E(z)\Delta x$$

The change in magnetic flux gives

$$-\frac{d\Phi_B}{dt} = -\frac{\partial}{\partial t}(B\Delta z\Delta x) = -\Delta z\Delta x\frac{\partial B}{\partial t}.$$

When we take limits of the deltas going to zero, we shrink down the box and our B will be B(z) pointing parallel to the y axis. So E and B will be perpendicular as we expect from earlier discussions. Combining the two equations, we arrive at

$$E(z+\Delta z)\Delta x - E(z)\Delta x = -\Delta z\Delta x\frac{\partial B}{\partial t}$$

Now comes a super cool part. We divide both sides by the delta-product area.

$$\frac{E(z+\Delta z) - E(z)}{\Delta z} = -\frac{\partial B}{\partial t}$$

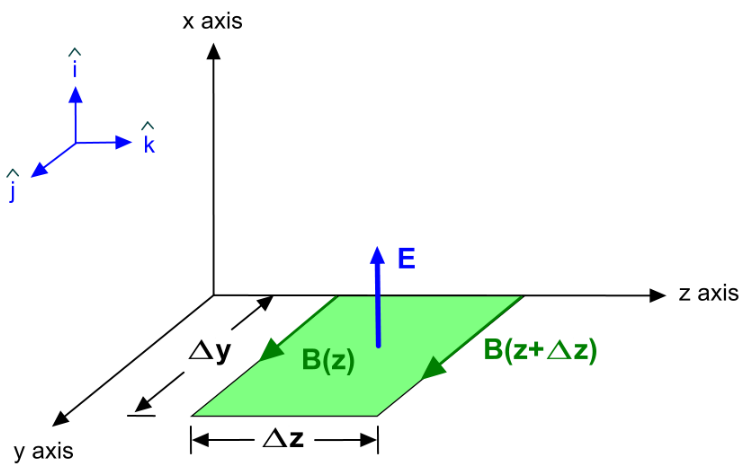
The limit as delta z goes to zero is understood. This is the derivative $\partial E / \partial z$.

So we have

$$\frac{\partial E}{\partial z} = -\frac{\partial B}{\partial t}$$

Next we do the same with the magnetic field using $\oint \vec{B} \cdot \vec{dl} = \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt}$.

The B field points along the y axis as we found earlier. Our new loop is in the y-z plane. We sketch the E flux that will result from the loop as an E field pointing upward, normal to our new loop. We do this because the normal vector of the new square loop has a unit vector pointing upward. If E turns out to be negative, a minus will appear in the calculation.



For the loop integral we need to go counterclockwise so that we are consistent with the E field and normal vectors that point upward. We find

$$\oint \vec{B} \cdot \vec{dl} = B(z)\Delta y - B(z+\Delta z)\Delta y = \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt} = \mu_0 \varepsilon_0 \frac{\partial}{\partial t}(E \Delta y \Delta z)$$

Dividing by the deltas, similar to what we did before, we obtain

$$-\left[\frac{B(z+\Delta z) - B(z)}{\Delta z}\right] = \mu_0 \varepsilon_0 \frac{\partial E}{\partial t} \text{, i.e., } -\frac{\partial B}{\partial z} = \mu_0 \varepsilon_0 \frac{\partial E}{\partial t}$$

since taking the limit is understood. We let this be understood so that our notation is easier to work with. Our results are

$$\frac{\partial E}{\partial z} = -\frac{\partial B}{\partial t} \quad \text{and} \quad \frac{\partial B}{\partial z} = -\mu_0 \varepsilon_0 \frac{\partial E}{\partial t}.$$

Since the wave equation is a differential equation with a second derivative, we take another spatial derivative to see where it leads us. We leave the second equation alone.

$$\frac{\partial^2 E}{\partial z^2} = -\frac{\partial}{\partial z}\frac{\partial B}{\partial t} \qquad \frac{\partial B}{\partial z} = -\mu_0 \varepsilon_0 \frac{\partial E}{\partial t}$$

The order of differentiation does not matter as z and t are independent of each other.

$$\frac{\partial^2 E}{\partial z^2} = -\frac{\partial}{\partial z}\frac{\partial B}{\partial t} = -\frac{\partial}{\partial t}\frac{\partial B}{\partial z}$$

Now substitute for the time derivative using the second of our starting equations.

$$\frac{\partial^2 E}{\partial z^2} = -\frac{\partial}{\partial t}\left[-\mu_0 \varepsilon_0 \frac{\partial E}{\partial t}\right]$$

This gives us.

$$\boxed{\frac{\partial^2 E}{\partial z^2} = \mu_0 \varepsilon_0 \frac{\partial^2 E}{\partial t^2}}$$

Compare this to the wave equation. $\boxed{\frac{\partial^2 \psi}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 \psi}{\partial t^2}}$

We have a wave equation with speed $c = \dfrac{1}{\sqrt{\mu_0 \varepsilon_0}}$. This is the speed of light.

PJ1 (Practice Problem). Show that $\dfrac{\partial^2 B}{\partial z^2} = \mu_0 \varepsilon_0 \dfrac{\partial^2 B}{\partial t^2}$.

One Final Important Observation.

In our wave equation for light, $\nabla^2\psi = \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}$. Let's look at this in one spatial dimension.

$$\frac{\partial^2\psi}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}$$

Write this as

$$\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} - \frac{\partial^2\psi}{\partial x^2} = 0$$

The space and time derivatives appear with the opposite sign and the "c" goes with the time as in relativity.

This equation appears the same under Lorentz transformation, where the x and t become prime but the speed of light stays the same. And this wave-equation result comes from the Maxwell equations. We again see the deep connection between relativity and electromagnetic theory.

The seeds of special relativity are hidden in the Maxwell equations. We have already seen that the very existence of the magnetic field can be arrived at from Coulomb's Law and special relativity. So electromagnetic theory contains special relativity!