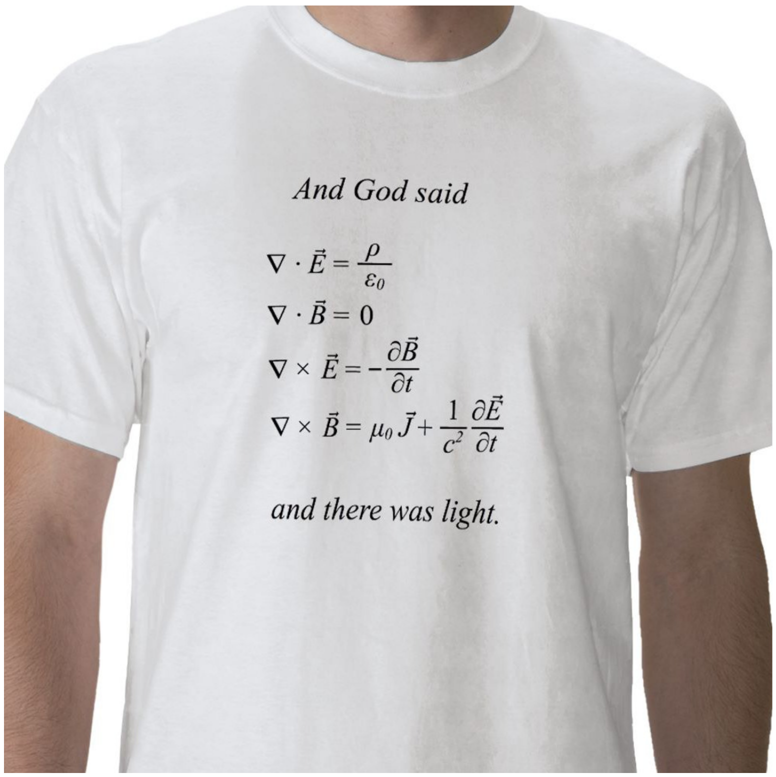
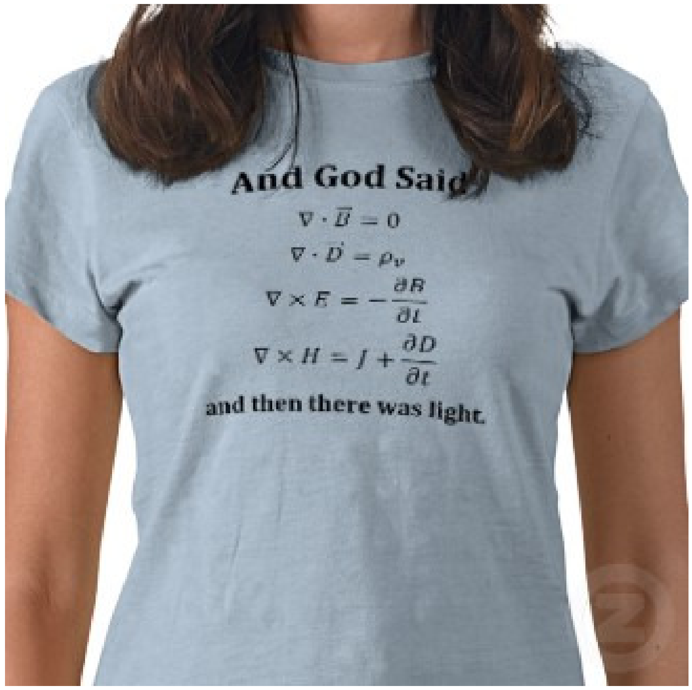
Electromagnetic Theory
Prof. Ruiz, UNC Asheville, doctorphys on YouTube
Chapter K Notes. Electromagnetic Waves

K1. "Let There Be Light." We have already seen a short derivation of the wave equation from the Maxwell equations. So we get light from the laws of electricity and magnetism. See the shirt below. Note that the replacement $\mu_0 \varepsilon_0 = \dfrac{1}{c^2}$ has been made in the fourth equation.



Courtesy www.zazzle.ca for Custom T-Shirts

The above shirt gives the Maxwell equations in the MKS system. This is the system we are using in this course. We can call this Dad's shirt. The next photo shows a shirt with the Maxwell equations in the CGS system and one where D replaces E and H replaces B. We will introduce D and H later in our course. We can call this Mom's shirt. The last one is Baby's shirt. Which system is used for the equations on baby's shirt?



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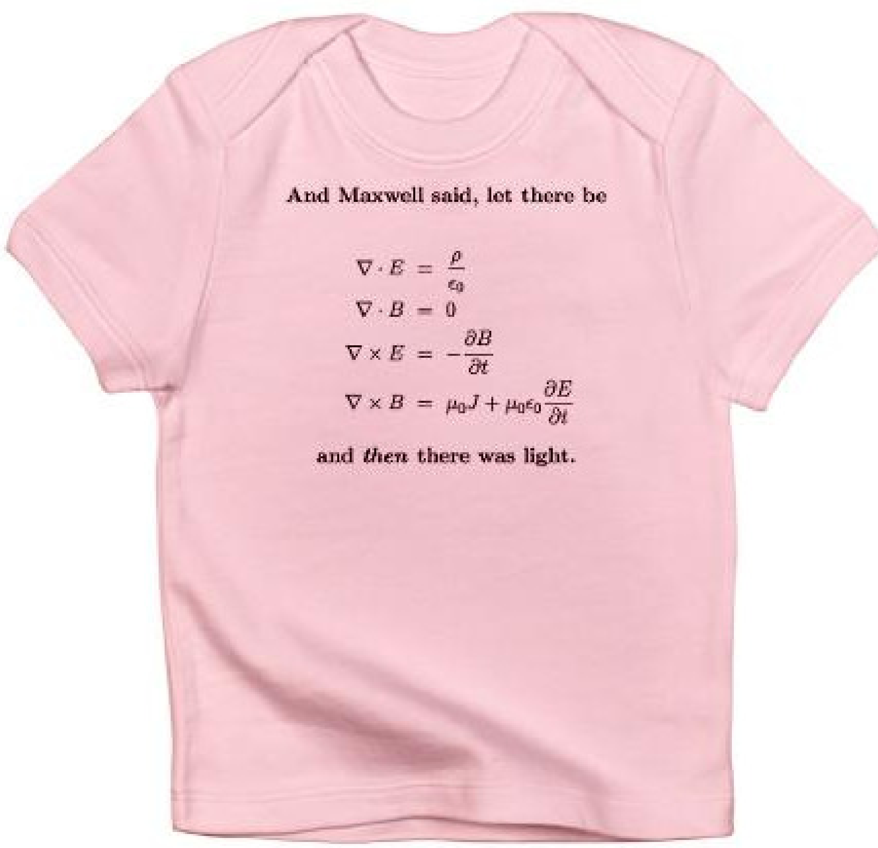


Courtesy www.cafepress.com

We now proceed to solve the Maxwell equations in free space and derive the wave equation formally from the differential version of the equations.

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$
$$\nabla \cdot \vec{B} = 0$$
$$\nabla \times B = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$
$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

Free-Space Equations

$$\nabla \cdot \vec{E} = 0$$
$$\nabla \cdot \vec{B} = 0$$
$$\nabla \times \vec{B} = \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$
$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

For the free space Maxwell equations we are far away from any charge sources and currents. Thus, we set $\rho = 0$ and $\vec{J} = 0$.

The free-space equations have beautiful symmetry and contain the secret about light. We play with these equations to see if a wave equation is supported. This is an example of theoretical physics at its best. We are in search of a discovery using theory only.

We are in search for a second order differential equation so we go for a second derivative with respect to time. Take a derivative of the equation $\nabla \times \vec{B} = \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$ with respect to time.

$$\frac{\partial}{\partial t}(\nabla \times \vec{B}) = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\nabla \times \frac{\partial \vec{B}}{\partial t} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

Now it's time to use the Maxwell equation with the $\frac{\partial \vec{B}}{\partial t}$, i.e., $(\nabla \times \vec{E}) = -\frac{\partial \vec{B}}{\partial t}$,

Write this last equation with the time-derivative first. Our two equations so far are then

$$\nabla\times\frac{\partial\vec{B}}{\partial t}=\mu_0\varepsilon_0\frac{\partial^2\vec{E}}{\partial t^2} \quad \text{and} \quad \frac{\partial\vec{B}}{\partial t}=-\nabla\times\vec{E}\,.$$

Substituting this second equation into our first equation gives us an equation with just E.

$$\nabla\times(-\nabla\times\vec{E})=\mu_0\varepsilon_0\frac{\partial^2\vec{E}}{\partial t^2}$$

$$\nabla\times(\nabla\times\vec{E})=-\mu_0\varepsilon_0\frac{\partial^2\vec{E}}{\partial t^2}$$

Let's focus on $\nabla\times(\nabla\times\vec{E})$. We do this by first calculating the curl of $\vec{E}$.

$$\nabla\times\vec{E}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ E_x & E_y & E_z\end{vmatrix}=\hat{i}\left(\frac{\partial E_z}{\partial y}-\frac{\partial E_y}{\partial z}\right)-\hat{j}\left(\frac{\partial E_z}{\partial x}-\frac{\partial E_x}{\partial z}\right)+\hat{k}\left(\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}\right)$$

Then

$$\nabla\times(\nabla\times\vec{E})=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ \frac{\partial E_z}{\partial y}-\frac{\partial E_y}{\partial z} & \frac{\partial E_x}{\partial z}-\frac{\partial E_z}{\partial x} & \frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}\end{vmatrix}.$$

Let's do the x-component first.

$$\nabla\times(\nabla\times\vec{E})_x=\frac{\partial}{\partial y}\left[\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}\right]-\frac{\partial}{\partial z}\left[\frac{\partial E_x}{\partial z}-\frac{\partial E_z}{\partial x}\right]$$

$$\nabla\times(\nabla\times\vec{E})_x=\frac{\partial^2 E_y}{\partial y\partial x}-\frac{\partial^2 E_x}{\partial y^2}-\frac{\partial^2 E_x}{\partial z^2}+\frac{\partial^2 E_z}{\partial z\partial x}$$

Flip the order of the derivatives for the first and last term to obtain

$$\nabla\times(\nabla\times\vec{E})_x=\frac{\partial^2 E_y}{\partial x\partial y}-\frac{\partial^2 E_x}{\partial y^2}-\frac{\partial^2 E_x}{\partial z^2}+\frac{\partial^2 E_z}{\partial x\partial z}.$$

$$\nabla\times(\nabla\times\vec{E})_x=\frac{\partial^2 E_y}{\partial x\partial y}+\frac{\partial^2 E_z}{\partial x\partial z}-\frac{\partial^2 E_x}{\partial y^2}-\frac{\partial^2 E_x}{\partial z^2}$$

$$\nabla\times(\nabla\times\vec{E})_x=\frac{\partial}{\partial x}\left[\frac{\partial E_y}{\partial y}+\frac{\partial E_z}{\partial z}\right]-\frac{\partial^2 E_x}{\partial y^2}-\frac{\partial^2 E_x}{\partial z^2}$$

We now add to the right side zero in the form of $\frac{\partial^2 E_x}{\partial x^2}-\frac{\partial^2 E_x}{\partial x^2}$:

$$\nabla\times(\nabla\times\vec{E})_x=\frac{\partial}{\partial x}\left[\frac{\partial E_x}{\partial x}+\frac{\partial E_y}{\partial y}+\frac{\partial E_z}{\partial z}\right]-\frac{\partial^2 E_x}{\partial x^2}-\frac{\partial^2 E_x}{\partial y^2}-\frac{\partial^2 E_x}{\partial z^2}.$$

$$\nabla\times(\nabla\times\vec{E})_x=\frac{\partial}{\partial x}\left[\nabla\cdot\vec{E}\right]-\nabla^2 E_x$$

Note that we have discovered the following powerful identity:

$$\nabla\times(\nabla\times\vec{E})=\nabla(\nabla\cdot\vec{E})-\nabla^2\vec{E}$$

But $\nabla\cdot\vec{E}=0$ in free space.

Therefore: $\nabla\times(\nabla\times\vec{E}) = \nabla(\nabla\cdot\vec{E}) - \nabla^2\vec{E}$ reduces to

$$\nabla\times(\nabla\times\vec{E}) = -\nabla^2\vec{E}\,.$$

Putting it all to together, our equation

$$\nabla\times(\nabla\times\vec{E}) = -\mu_0\varepsilon_0\frac{\partial^2\vec{E}}{\partial t^2} \text{ becomes } \nabla^2\vec{E} = \mu_0\varepsilon_0\frac{\partial^2\vec{E}}{\partial t^2}\,.$$

Voilà! Compare this equation $\nabla^2\vec{E} = \mu_0\varepsilon_0\frac{\partial^2\vec{E}}{\partial t^2}$ to the wave equation

$$\frac{\partial\psi^2(x,t)}{\partial x^2} = \frac{1}{v^2}\frac{\partial\psi^2(x,t)}{\partial t^2}$$

It is the wave equation for the electric field with $\frac{1}{v^2} = \mu_0\varepsilon_0$.

Guess what Maxwell found for the speed $v = \frac{1}{\sqrt{\mu_0\varepsilon_0}}$ when he put in the numerical values for μ_0 and ε_0? He found a value close to the then known value of the speed of light. This was in 1861. He concluded that light was an electromagnetic phenomenon. We will summarize our results below replacing the speed with the speed of light symbol.

$$\nabla^2\vec{E} = \mu_0\varepsilon_0\frac{\partial^2\vec{E}}{\partial t^2}\text{, where } c = \frac{1}{\sqrt{\mu_0\varepsilon_0}}\,.$$

Therefore,

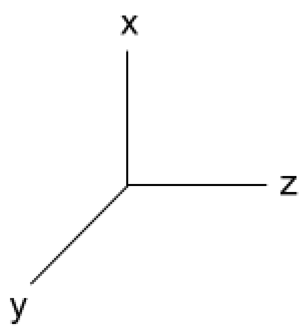
$$\boxed{\nabla^2\vec{E} = \frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2}}$$

For homework, you are going to derive the same equation, but with the magnetic field replacing the electric field. It will be fast because you will be coaxed to use the powerful identity we derived, thus taking a shortcut.

$$\boxed{\nabla^2 \vec{B} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2}}.$$

Once again we find $c = \dfrac{1}{\sqrt{\mu_0 \varepsilon_0}}$.

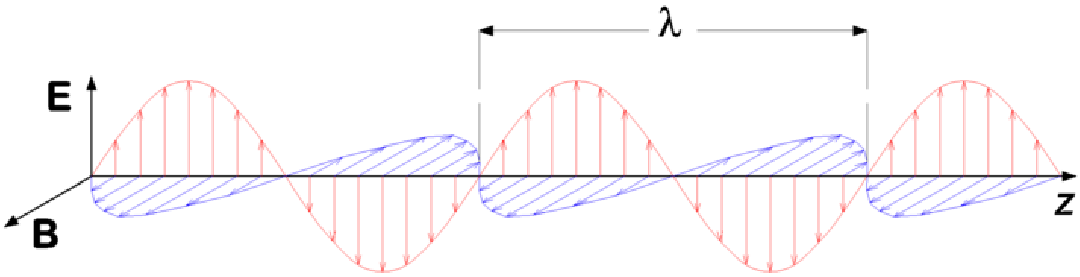
K2. Electromagnetic Waves



The axes at the left are defined with the usual association of the unit vectors $\hat{i}$, $\hat{j}$, and $\hat{k}$ with x, y, and z respectively. Note also that we have a right-handed system with

$$\hat{i} \times \hat{j} = \hat{k} \, .$$

For $\vec{E} = E_0 \sin\left[k(z - ct)\right] \hat{i}$, you will show for homework that $\vec{B}$ is along the y axis with $\vec{B} = B_0 \sin\left[k(z - ct)\right] \hat{j}$, i.e., in phase with $\vec{E}$.



Courtesy P.wormer, Wikimedia

Below we discuss some properties of waves.

1. The Wavelength λ - freezing the wave in time. We can take t = 0 as our fixed point. Let's analyze in general for wave where the wave speed is v. Then we use

$\sin[k(z-vt)]$. For t = 0, we simply have $\sin kz$. The wave repeats in space every time the argument in the sine function advances by 2π. The distance in space corresponding to this is called the wavelength λ.

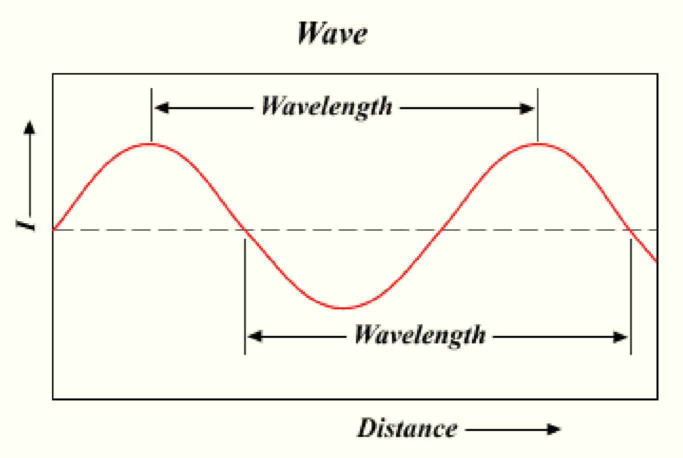


Figure Courtesy Bryan Derksen and Dicklyon/Eptalon, Wikipedia

So we set $kz = 2\pi$, then z is our wavelength.

$$k\lambda = 2\pi$$

Our constant k is called the wave number.

$$k = \frac{2\pi}{\lambda}$$

However, k has units of inverse meters, i.e., inverse distance. But it does tell you how many wavelengths correspond to the angular measure 2π.

2. The Period T - freezing yourself to stay at the same point in space. We can take x = 0 as our free point. Then $\sin[k(z-vt)]$ becomes $\sin(-kvt) = -\sin kvt$. The wave repeats its pattern at the fixed location every time the argument in the sine function advances by 2π. The time interval corresponding to this is called the period T. So we set $kvt = 2\pi$, then t is our period.

$$kvT = 2\pi$$

Substituting $k = \frac{2\pi}{\lambda}$ in this equation we obtain

$$\frac{2\pi}{\lambda} vT = 2\pi$$

$$\lambda = vT$$

This is a form of the equation distance is equal to speed times time: d = vt.

3. The Frequency f - the frequency is how many times the wave goes through its pattern per second, i.e., the number of "wiggles per second." Think of the crest as the

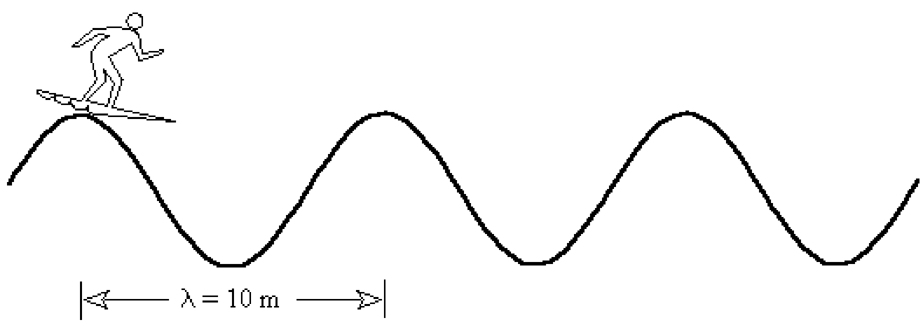
"wig" and the trough as the "gle." If it takes you 1/4 second to do one pattern, then your frequency is 4 1/s, i.e., 4 per second. The unit for per second is hertz, written as 1/s = Hz.

Therefore $f = \frac{1}{T}$ and $T = \frac{1}{f}$. We express $\lambda = vT$ in terms of the frequency. We get the following equations.

$$\lambda = \frac{v}{f}$$

$$v = \lambda f$$

Here is a quick way to see this. Imagine a perfect surfer and wave where the crests are 10 m apart and say 5 go by per second. The speed is then 50 m/s.



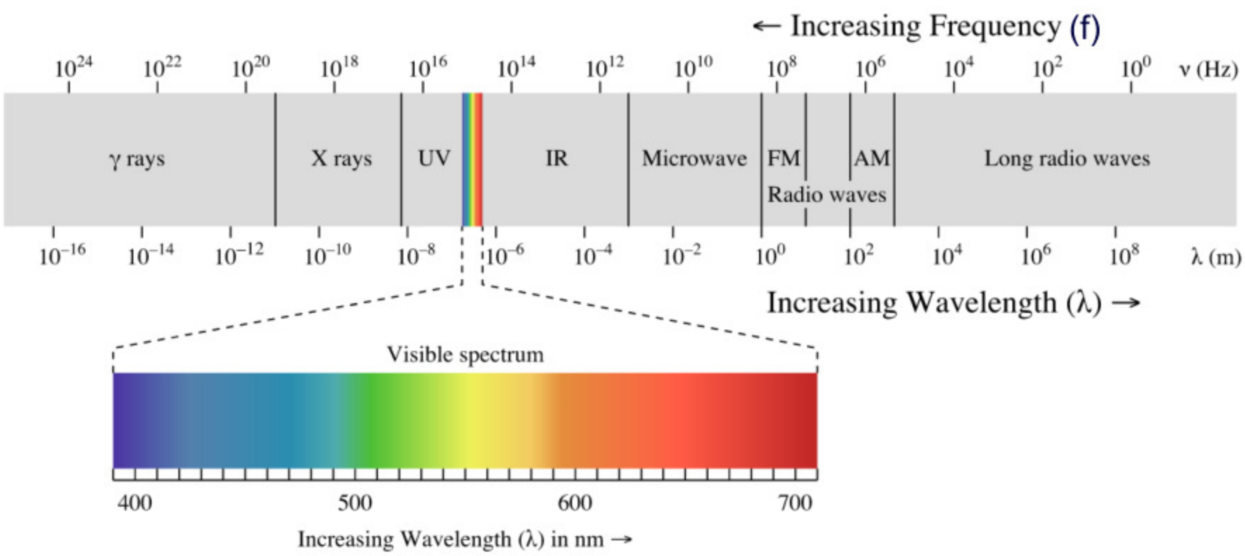
The values for the parameters are: $\lambda = 10\,m$, $f = 5\,Hz$, and $v = 50\,\frac{m}{s}$.

$$v = \lambda f$$

Now getting back to light waves we can write

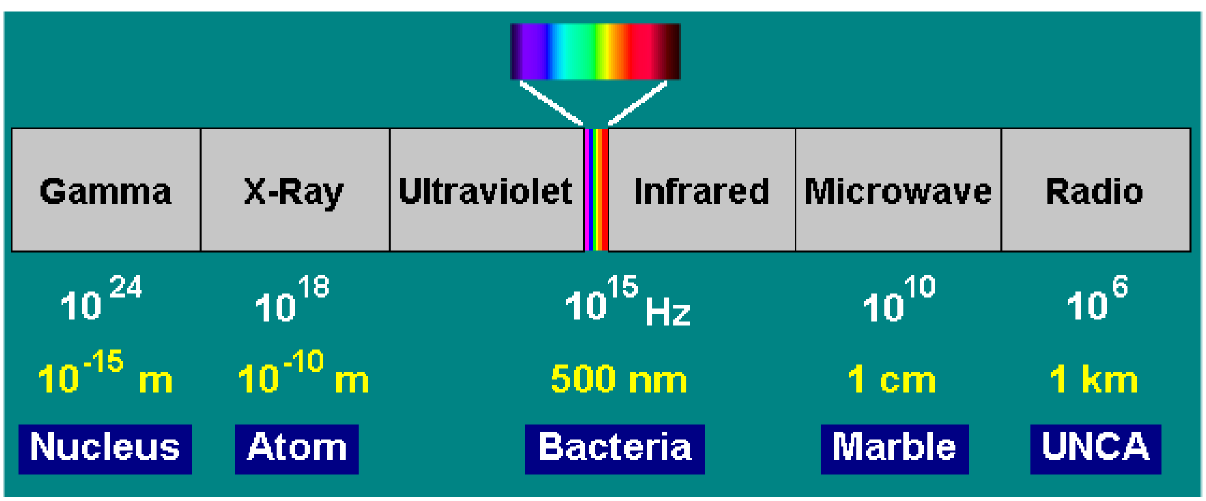
$$c = \lambda f\ .$$

You can initiate an electric field disturbance by shaking a charged particle. In this way you can understand all the forms of electromagnetic radiation. Or you can shake a magnet. The next figure lists the various type of electromagnetic radiation. Each can be referenced by its wavelength or its frequency.



Adapted from Philip Ronan, Wikimedia

The visible spectrum makes up only a small part of the entire electromagnetic radiation. Below is a summary diagram giving some frequencies in Hz and wavelengths with comparable sizes of the wavelengths. UNCA = UNC Asheville (the campus).



So Electromagnetic Theory gives birth to Optics!