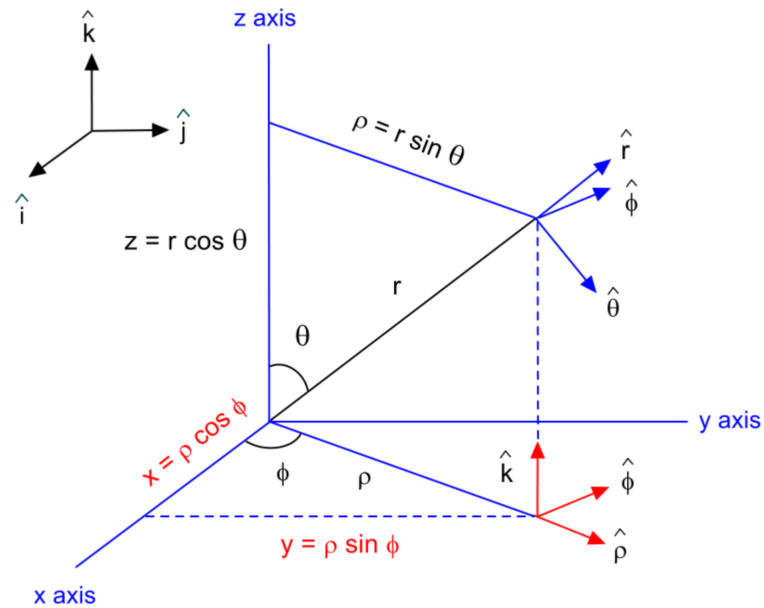
Electromagnetic Theory
Prof. Ruiz, UNC Asheville, doctorphys on YouTube
Chapter L Notes. Curvilinear Coordinates

L1. Curvilinear Coordinates

Cartesian (x, y, z) Cylindrical (ρ, ϕ, z) Spherical (r, θ, ϕ)



Cylindrical to Cartesian

$$x = \rho \cos\phi$$
$$y = \rho \sin\phi$$
$$z = z$$

Spherical to Cartesian

$$x = r \sin\theta \cos\phi$$
$$y = r \sin\theta \sin\phi$$
$$z = r \cos\theta$$

Cartesian to Cylindrical: $\rho = \sqrt{x^2 + y^2}$ $\quad \tan\phi = \dfrac{y}{x}$ $\quad z = z$

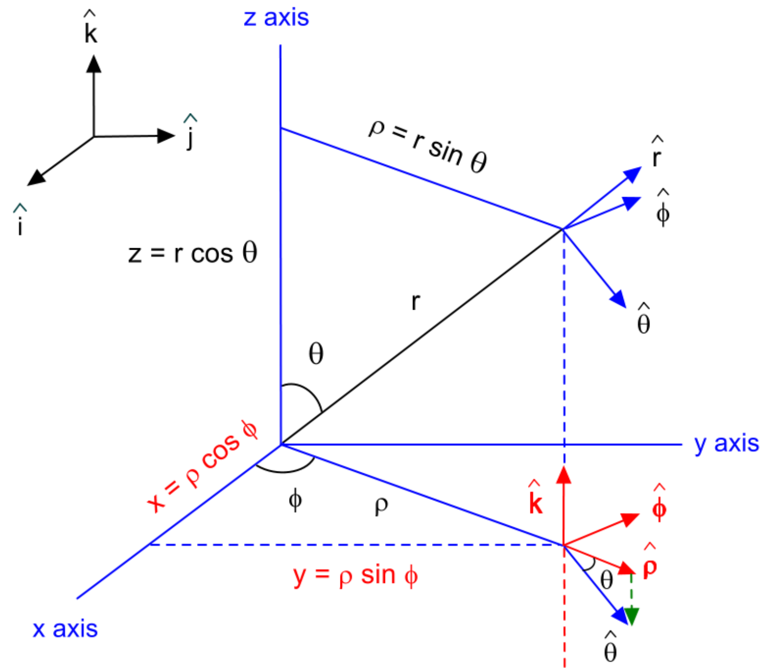
Cartesian to Spherical: $r = \sqrt{x^2 + y^2 + z^2}$ $\quad \tan\phi = \dfrac{y}{x}$ $\quad \cos\theta = \dfrac{z}{r}$

Unit Vectors: $\hat{\rho} = \cos\phi\, \hat{i} + \sin\phi\, \hat{j}$ $\quad \hat{\phi} = \hat{k} \times \hat{\rho} = \cos\phi\, \hat{j} - \sin\phi\, \hat{i}$

$$\hat{r} = \sin\theta\, \hat{\rho} + \cos\theta\, \hat{k}$$

PL1 (Practice Problem). Use $\hat{\theta} = \hat{\phi} \times \hat{r}$ to show that $\hat{\theta} = \cos\theta\, \hat{\rho} - \sin\theta\, \hat{k}$.

PL2 (Practice Problem). Arrive at the result $\hat{\theta} = \cos\theta\, \hat{\rho} - \sin\theta\, \hat{k}$ by inspecting the figure on the next page. Do not take any cross products. Simply write the answer down by clever inspection of the figure, where we have moved $\hat{\theta}$ down carefully without changing its direction or length.



Unit-Vector Summary:

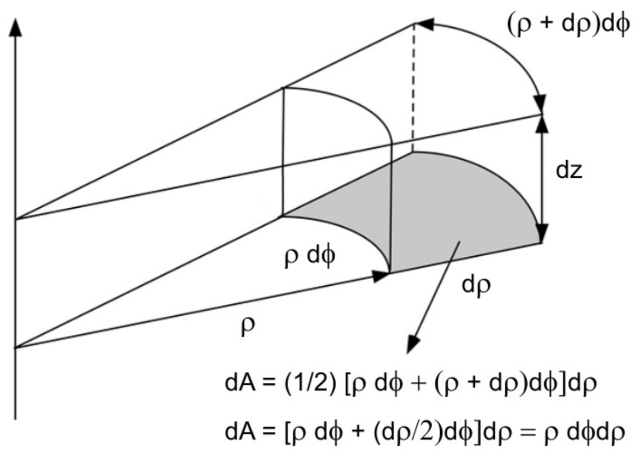
$$\hat{\rho} = \cos\phi\,\hat{i} + \sin\phi\,\hat{j}$$

$$\hat{\phi} = -\sin\phi\,\hat{i} + \cos\phi\,\hat{j}$$

$$\hat{r} = \sin\theta\,\hat{\rho} + \cos\theta\,\hat{k}$$

$$\hat{\theta} = \cos\theta\,\hat{\rho} - \sin\theta\,\hat{k}$$

Cylindrical Coordinates Differential Line and Volume Elements. Figure from Tony Saad.



For the volume element we write $dV = dAdz$, where the dA is the area of the base in the figure. This is simply $dA = \rho d\phi d\rho$. But if you want to be super careful, use the average of the shorter and longer arc lengths (see left). As we take the limit to the infinitesimal, we a product of three differentials will vanish faster than the product of two. Therefore, we can toss the $d\rho d\phi d\rho$ term compared to the $d\phi d\rho$ term. The volume element is then

$$dV = \rho d\phi d\rho dz.$$

The line element for the diagonal of the "cube-like" region is found by using the Pythagorean theorem twice: once for the floor $d\rho^2 + \rho^2 d\phi^2$ and then the diagonal of the floor with the height dz.

$$ds^2 = d\rho^2 + \rho^2 d\phi^2 + dz^2$$

PL3 (Practice Problem with Solution). To verify that we can use length times width times height for curved boundaries let's integrate out the volume of a can this way.

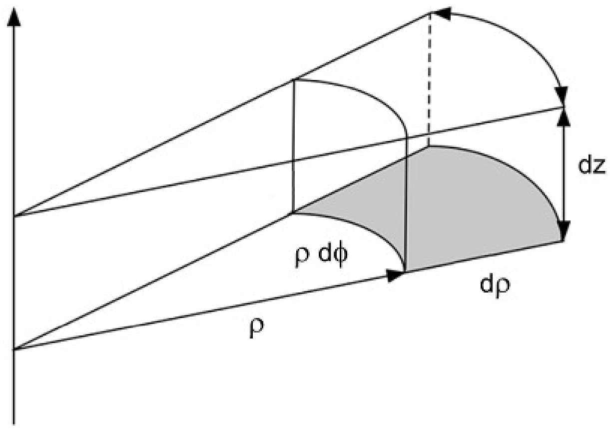


Figure from Tony Saad.

Let the can have radius R and height h. We want to check out

$$dV = \rho d\phi d\rho dz$$

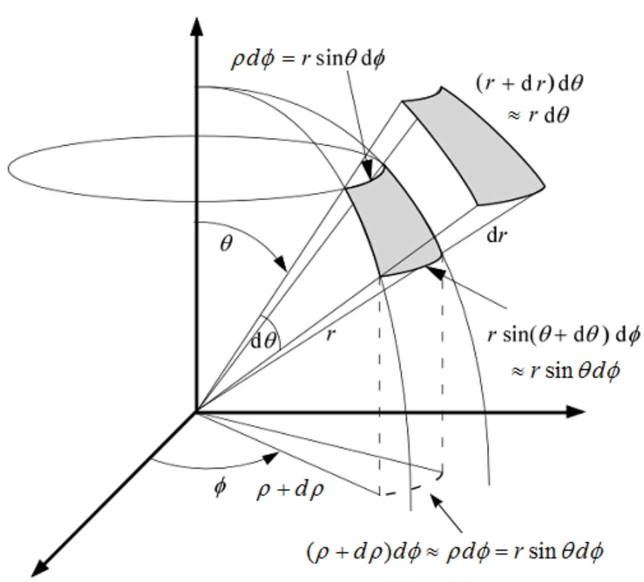
to see if it integrates out to the right volume.

$$dV = \int_0^R \rho d\rho \int_0^{2\pi} d\phi \int_0^h dz$$

$$dV = \frac{\rho^2}{2}\Big|_0^R \phi\Big|_0^{2\pi} z\Big|_0^h = \frac{R^2}{2} 2\pi h = \pi R^2 h$$

This is the volume of a cylinder. So we are good.

Spherical Coordinates Differential Line and Volume Elements. Figure from Tony Saad.



The volume element is found using length times width times height for the tilted volume element.

$$dV = (r\sin\theta d\phi)(rd\theta)dr$$

We don't even take the average of the widths since to first order, the differential

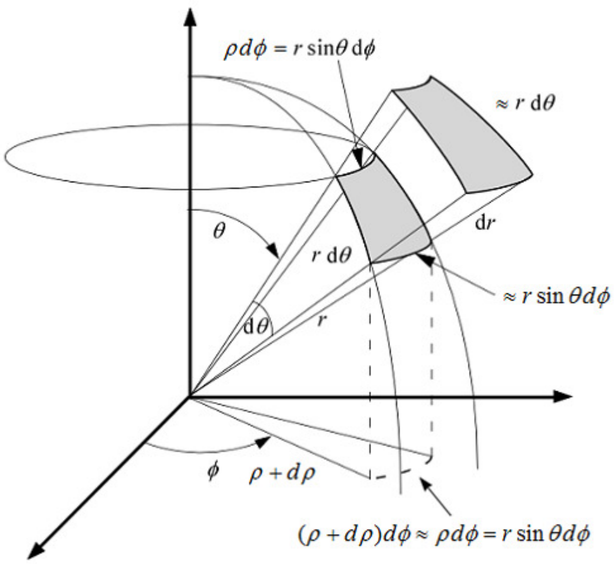
$$(r\sin\theta + d\theta)d\phi$$

becomes $r\sin\theta d\phi$.

The volume element is usually written as

$$dV = r^2 \sin\theta dr d\theta d\phi \,.$$

The differential line element is the square of the diagonal as before. So we use simple square each of the differential sides. $ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2$



PL4 (Practice Problem with Solution). Let's see if we get the volume of a sphere with radius R, using the volume formula for a rectangular solid even though our boundary lines are curved.

$$dV = r^2 \sin\theta dr d\theta d\phi$$

$$dV = \int_0^R r^2 dr \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\phi$$

$$dV = \frac{r^3}{3}\bigg|_0^R \left[-\cos\theta\right]_0^\pi \phi\Big|_0^{2\pi} = \frac{R^3}{3}(-1)\left[-1-1\right]2\pi = \frac{R^3}{3}2\cdot 2\pi = \frac{4}{3}\pi R^3$$

This checks out. So we are okay with how we are setting things up.

Here is the summary for the differential elements for all three coordinate systems.

Cartesian:

$$dV = dxdydz \qquad ds^2 = dx^2 + dy^2 + dz^2$$

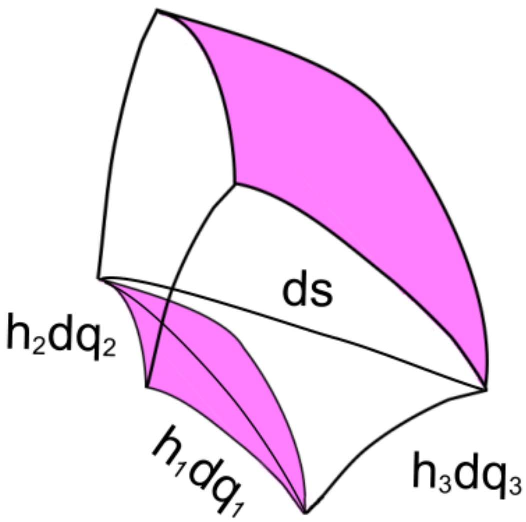
Cylindrical:

$$dV = \rho\, d\rho d\phi dz \qquad ds^2 = d\rho^2 + \rho^2 d\phi^2 + dz^2$$

Spherical:

$$dV = r^2 \sin\theta dr d\theta d\phi \qquad ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2$$

We next consider an infinitesimal solid in general curvilinear coordinates. Note that each variable differential has a factor in front. These "h" factors are called scale factors. Familiar examples are the arc lengths we just investigated for cylindrical and spherical coordinates: $ds = r d\theta$ or $ds = r \sin\theta d\phi$.



Differential Line Element. We use the Pythagorean theorem twice for the diagonal of the solid. First, for the base, we have for the square of the hypotenuse

$$h_1^2 dq_1^2 + h_2^2 dq_2^2 .$$

Second, adding to this the square of the height gives

$$ds^2 = h_1^2 dq_1^2 + h_2^2 dq_2^2 + h_3^2 dq_3^2 .$$

Differential Volume Element. Here we use the length times width times height idea to arrive at the volume element.

$$dV = (h_1 dq_1)(h_2 dq_2)(h_3 dq_3)$$

$$dV = h_1 h_2 h_3 dq_1 dq_2 dq_3$$

General Case: $dV = h_1 h_2 h_3 dq_1 dq_2 dq_3$ and $ds^2 = h_1^2 dq_1^2 + h_2^2 dq_2^2 + h_3^2 dq_3^2$.

Using the coordinates and scale factors for each of our coordinate systems, we can find dV and ds^2 from the above master formulas.

Cartesian: $q_1 = x$, $q_2 = y$, $q_3 = z$ with $h_1 = 1$, $h_2 = 1$, $h_3 = 1$

Cylindrical: $q_1 = \rho$, $q_2 = \phi$, $q_3 = z$ with $h_1 = 1$, $h_2 = \rho$, $h_3 = 1$

Spherical: $q_1 = r$, $q_2 = \theta$, $q_3 = \phi$ with $h_1 = 1$, $h_2 = r$, $h_3 = r \sin\theta$

Scale Factors are in general functions. That makes coordinates curvilinear.

L2. The Gradient. The Gradient in Cartesian coordinates is given by

$$\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}.$$

We can obtain the gradient in curvilinear coordinates by comparing the differentials. In Cartesian coordinates we have these three differentials:

dx, dy, and dz.

In cylindrical coordinates we have: $d\rho$, $\rho d\phi$, and dz. Note that each has dimension of length. Our gradient in cylindrical coordinates is then

$$\nabla f = \frac{\partial f}{\partial \rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial f}{\partial \phi}\hat{\phi} + \frac{\partial f}{\partial z}\hat{k}.$$

In spherical coordinates we have: dr, $rd\theta$, and $r\sin\theta d\phi$. Note that each has dimension of length. Our gradient in spherical coordinates is then

$$\nabla f = \frac{\partial f}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\hat{\phi}.$$

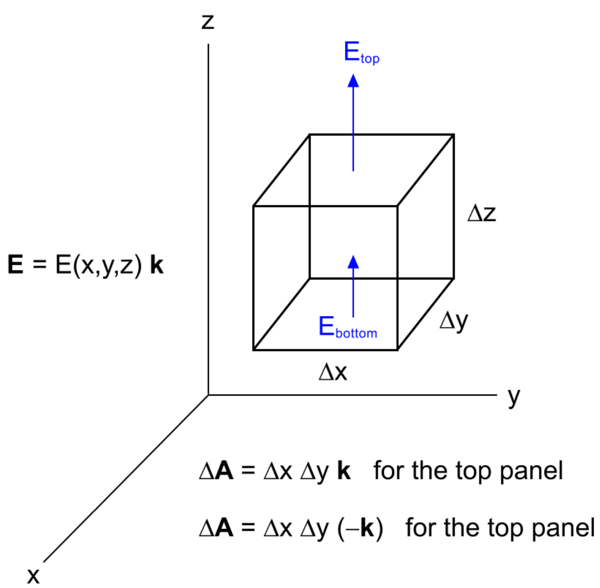
In curvilinear coordinates we have: $h_1 dq_1$, $h_2 dq_2$, and $h_3 dq_3$. Note that each has dimension of length. Our gradient in curvilinear coordinates is then

$$\nabla f = \frac{1}{h_1}\frac{\partial f}{\partial q_1}\hat{e}_1 + \frac{1}{h_2}\frac{\partial f}{\partial q_2}\hat{e}_2 + \frac{1}{h_3}\frac{\partial f}{\partial q_3}\hat{e}_3.$$

Caution: If you write the operator by itself, put the unit vectors on the left since they are in general functions of the coordinates.

$$\nabla = \hat{e}_1\frac{1}{h_1}\frac{\partial}{\partial q_1} + \hat{e}_2\frac{1}{h_2}\frac{\partial}{\partial q_2} + \hat{e}_3\frac{1}{h_3}\frac{\partial}{\partial q_3}$$

L3. The Divergence Theorem. Here is a quick review of the derivation of the divergence theorem in Cartesian coordinates.



We are interested in calculating the flux through the enclosed surface:

$$\oiint \vec{E}\cdot\overrightarrow{dA}\,.$$

$E_{bottom} = E_z(x,y,z)$ and its counterpart $E_{top} = E_z(x,y,z+\Delta z)$.

The net flux out of the surface of our cube is given as follows.

$$\oiint \vec{E}\cdot\overrightarrow{dA} => E_z(x,y,z+\Delta z)\Delta x\Delta y - E_z(x,y,z)\Delta x\Delta y$$

$$\oiint \vec{E}\cdot\overrightarrow{dA} => \frac{E_z(x,y,z+\Delta z)-E_z(x,y,z)}{\Delta z}\Delta x\Delta y\Delta z$$

$$\oiint \vec{E}\cdot\overrightarrow{dA} = \iiint \frac{\partial E_z}{\partial z}\,dxdydz$$

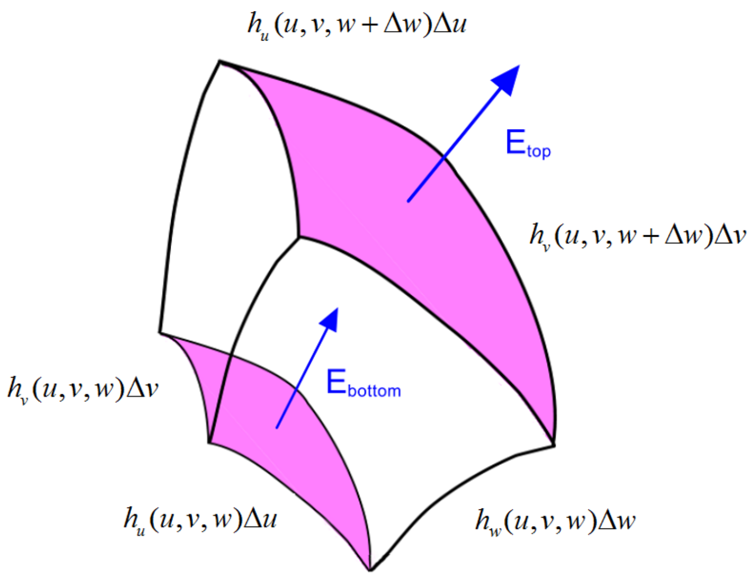
$$\oiint \vec{E}\cdot\overrightarrow{dA} = \iiint \left[\frac{\partial E_x}{\partial x}+\frac{\partial E_y}{\partial y}+\frac{\partial E_z}{\partial z}\right]dxdydz$$

$$\oiint \vec{E}\cdot\overrightarrow{dA} = \iiint \left[\nabla\cdot\vec{E}\right]dxdydz$$

PL5 (Practice Problem). Show that starting we $E_{bottom} = E_z(x+\frac{\Delta x}{2}, y+\frac{\Delta y}{2}, z)$ and its appropriate counterpart that you get the same result. You will find that taking limits as the extra deltas go to zero do not lead to any derivatives. They just go away.

L3. The Divergence Theorem in Curvilinear Coordinates

We are going to redo our derivation of the divergence theorem in general now for curvilinear coordinates. We consider $(q_1, q_2, q_3) = (u, v, w)$ with scale factors (h_u, h_v, h_w). It is easier to work without subscripts for the coordinates.



The important thing to notice is that the scale factors are in general functions of the coordinates, i.e.,

$$h_w = h_w(u, v, w).$$

As a specific example, in spherical coordinates

$$h_\phi = h_\phi(r, \theta, \phi) = r \sin\theta .$$

As before, consider a simplified vector field such that

$$\vec{E}(u, v, w) = E_w \hat{e}_w .$$

We then subtract the flux out at the top from the flux in at the bottom along this dimension. Our enclosed integral $\oiint \vec{E} \cdot \vec{dA}$ will include different areas at the bottom and top.

$$dA_{bottom} = [h_u(u, v, w)\Delta u][h_v(u, v, w)\Delta v]$$

$$dA_{top} = [h_u(u, v, w+\Delta w)\Delta u][h_v(u, v, w+\Delta w)\Delta v]$$

Therefore, $\oiint \vec{E} \cdot \vec{dA} \Rightarrow$

$$E_w(u, v, w+\Delta w)[h_u(u, v, w+\Delta w)\Delta u][h_v(u, v, w+\Delta w)\Delta v]$$

$$-E_w(u, v, w)[h_u(u, v, w)\Delta u][h_v(u, v, w)\Delta v]$$

$$\oiint \vec{E} \cdot \vec{dA} \Rightarrow \left[(E_w h_u h_v)\big|_{w+\Delta w} - (E_w h_u h_v)\big|_w \right] \Delta u \Delta v$$

$$\oiint \vec{E} \cdot \overrightarrow{dA} => \left[\frac{(E_w h_u h_v)\big|_{w+\Delta w} - (E_w h_u h_v)\big|_w}{\Delta w} \right] \Delta u \Delta v \Delta w$$

$$\oiint \vec{E} \cdot \overrightarrow{dA} => \frac{\partial (E_w h_u h_v)}{\partial w} \Delta u \Delta v \Delta w$$

Now we want the differential volume element

$dV = (h_u \Delta u)(h_v \Delta v)(h_w \Delta w) = h_u h_v h_w \Delta u \Delta v \Delta w$ in there.

$$\oiint \vec{E} \cdot \overrightarrow{dA} => \frac{1}{h_u h_v h_w} \frac{\partial (E_w h_u h_v)}{\partial w} h_u h_v h_w \Delta u \Delta v \Delta w$$

Now it's time to put the q-variables back in.

$$\oiint \vec{E} \cdot \overrightarrow{dA} => \frac{1}{h_1 h_2 h_3} \frac{\partial (E_3 h_1 h_2)}{\partial w} h_1 h_2 h_3 \Delta q_1 \Delta q_2 \Delta q_3$$

$$\oiint \vec{E} \cdot \overrightarrow{dA} = \iiint \frac{1}{h_1 h_2 h_3} \frac{\partial (E_3 h_1 h_2)}{\partial q_3} dV$$

From this we can generalize to the general form of the divergence.

$$\nabla \cdot \vec{E} = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial (E_1 h_2 h_3)}{\partial q_1} + \frac{\partial (E_2 h_1 h_3)}{\partial q_2} + \frac{\partial (E_3 h_1 h_2)}{\partial q_3} \right]$$

PL6 (Practice Problem). Show that the divergence in curvilinear coordinates, i.e.,

$$\nabla \cdot \vec{A} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial(A_1 h_2 h_3)}{\partial q_1} + \frac{\partial(A_2 h_1 h_3)}{\partial q_2} + \frac{\partial(A_3 h_1 h_2)}{\partial q_3}\right],$$

reduces to the following in Cartesian, cylindrical, and spherical coordinates.

Cartesian: $$\nabla \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

Cylindrical: $$\nabla \cdot \vec{A} = \frac{1}{\rho}\frac{\partial(\rho A_\rho)}{\partial \rho} + \frac{1}{\rho}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

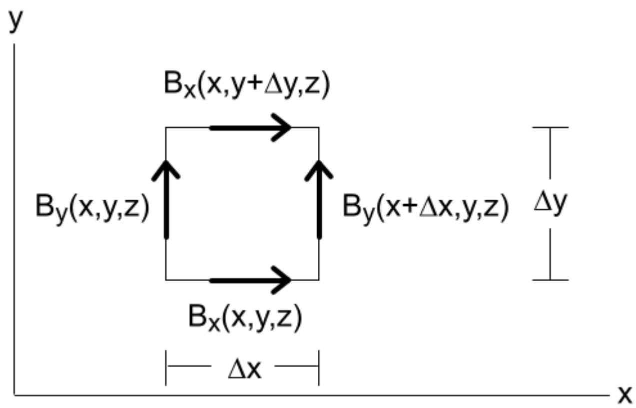
Spherical: $$\nabla \cdot \vec{A} = \frac{1}{r^2}\frac{\partial(r^2 A_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta A_\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi}$$

L4. Stoke's Theorem

In a previous chapter we considered a vector field **B** and proceed to do a closed line integral of this field.

$$\oint \vec{B} \cdot \vec{dl}$$

You recognize this as the left side of one of our Maxwell equations. The vector field can be any vector field. To simplify, we will pick the field to be in the x-y plane.



We obtained $\oint \vec{B} \cdot \vec{dl} =>$

$$\frac{B_y(x+\Delta x, y, z) - B_y(x, y, z)}{\Delta x} \Delta x \Delta y - \frac{B_x(x, y+\Delta y, z) - B_x(x, y, z)}{\Delta y} \Delta x \Delta y$$

PL7 (Practice Problem). Show where each piece above comes from.

We can extend this to curvilinear coordinates (u,v,w) with scale factors (h_u,h_v,h_w).

$$\oint \vec{B} \cdot \vec{dl} => \frac{\left[B_v(u,v,w)h_v(u,v,w)\right]_{u+\Delta u} - \left[B_v(u,v,w)h_v(u,v,w)\right]_u}{\Delta u} \Delta u \Delta v$$

$$-\frac{\left[B_u(u,v,w)h_u(u,v,w)\right]_{v+\Delta v} - \left[B_u(u,v,w)h_u(u,v,w)\right]_v}{\Delta v} \Delta u \Delta v$$

$$\oint \vec{B} \cdot \vec{dl} = \frac{1}{h_u h_v} \iint_A \left[\frac{\partial (B_v h_v)}{\partial u} - \frac{\partial (B_u h_u)}{\partial v} \right] dA_w$$

Introducing the q-variables $(q_1, q_2, q_3) = (u, v, w)$ with (h_1, h_2, h_3).

$$\oint \vec{B} \cdot \vec{dl} = \frac{1}{h_1 h_2} \iint_A \left[\frac{\partial(B_2 h_2)}{\partial q_1} - \frac{\partial(B_1 h_1)}{\partial q_2} \right] dA_3$$

The third component of the cross product defined as

$$(\nabla \times \vec{B})_3 = \frac{1}{h_1 h_2} \left[\frac{\partial(B_2 h_2)}{\partial q_1} - \frac{\partial(B_1 h_1)}{\partial q_2} \right].$$

leads to Stoke's Theorem in curvilinear coordinates.

$$\oint \vec{B} \cdot \vec{dl} = \iint_A (\nabla \times \vec{B}) \cdot \vec{dA}$$

Our cross product $\nabla \times \vec{B} = \frac{1}{h_1 h_2} \left[\frac{\partial(B_2 h_2)}{\partial q_1} - \frac{\partial(B_1 h_1)}{\partial q_2} \right] \hat{e}_3$ is better written as

$$\nabla \times \vec{B} = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial(B_2 h_2)}{\partial q_1} - \frac{\partial(B_1 h_1)}{\partial q_2} \right] h_3 \hat{e}_3$$

Then we can use the following determinant to express this.

$$\nabla \times \vec{B} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \hat{e}_1 & h_2 \hat{e}_2 & h_3 \hat{e}_3 \\ \frac{\partial}{\partial q_1} & \frac{\partial}{\partial q_2} & \frac{\partial}{\partial q_3} \\ h_1 B_1 & h_2 B_2 & h_3 B_3 \end{vmatrix}$$

L5. The Laplacian

We find the Laplacian of a function, i.e., $\nabla^2 f$, by applying the divergence to the gradient of a function: $\nabla^2 f = \nabla \cdot (\nabla f)$. We start with our previous result for the gradient

$$\nabla f = \frac{1}{h_1}\frac{\partial f}{\partial q_1}\hat{e}_1 + \frac{1}{h_2}\frac{\partial f}{\partial q_2}\hat{e}_2 + \frac{1}{h_3}\frac{\partial f}{\partial q_3}\hat{e}_3 .$$

and then use our previous result for the divergence

$$\nabla \cdot \vec{A} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial(A_1 h_2 h_3)}{\partial q_1} + \frac{\partial(A_2 h_1 h_3)}{\partial q_2} + \frac{\partial(A_3 h_1 h_2)}{\partial q_3}\right],$$

where $\vec{A} = \nabla f$. This means substituting

$$A_1 = \frac{1}{h_1}\frac{\partial f}{\partial q_1}, \quad A_2 = \frac{1}{h_2}\frac{\partial f}{\partial q_2}, \text{ and } A_3 = \frac{1}{h_3}\frac{\partial f}{\partial q_3} .$$

We obtain for $\nabla^2 f$ the following.

$$\frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial q_1}\left(\frac{1}{h_1}\frac{\partial f}{\partial q_1} h_2 h_3\right) + \frac{\partial}{\partial q_2}\left(\frac{1}{h_2}\frac{\partial f}{\partial q_2} h_1 h_3\right) + \frac{\partial}{\partial q_3}\left(\frac{1}{h_3}\frac{\partial f}{\partial q_3} h_1 h_2\right)\right]$$

This simplifies to

$$\nabla^2 f = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial q_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial f}{\partial q_1}\right) + \frac{\partial}{\partial q_2}\left(\frac{h_1 h_3}{h_2}\frac{\partial f}{\partial q_2}\right) + \frac{\partial}{\partial q_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial f}{\partial q_3}\right)\right]$$

PL8 (Practice Problem). Show our Laplacian in curvilinear coordinates, which is

$$\nabla^2 = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial q_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial}{\partial q_1}\right) + \frac{\partial}{\partial q_2}\left(\frac{h_1 h_3}{h_2}\frac{\partial}{\partial q_2}\right) + \frac{\partial}{\partial q_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial}{\partial q_3}\right)\right],$$

reduces to the following in Cartesian, cylindrical, and spherical coordinates.

Cartesian: $$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Cylindrical: $$\nabla^2 = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}$$

Spherical: $$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2}{\partial \phi^2}$$

Many years ago Mr. Samuel S. Ensor, my Calculus teacher at St. Joseph's College in Philadelphia (now university) gave us a project in Calculus III that was long, but very useful and productive (Spring 1969). It is given below. Everyone aspiring to be a physicist or engineer should do this calculation once in a lifetime. It will correct any rough edges you have in taking partial derivatives and using the chain rule.

PL9 (Practice Problem for the Summer). Derive the Laplacian in spherical coordinates the long way! Start with

$$x = r\sin\theta\cos\phi,\quad y = r\sin\theta\sin\phi,\quad z = r\cos\theta$$ **and**

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

Then you start cranking: $\frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta} + \frac{\partial \phi}{\partial x}\frac{\partial}{\partial \phi}$ and so on. Have fun!