Electromagnetic Theory
Prof. Ruiz, UNC Asheville, doctorphys on YouTube
Chapter M Notes. Laplace's Equation

M1. Review of Necessary Former Material

1. The Electric Potential. Recall in your study of mechanics the usefulness of the potential. Here is how it arises in the simplest case of constant gravity. Pick a mass up and move it upward a distance h. Then the work that the lifter must apply is

$$W = \int_0^h mgdz = mgh \,.$$

Holding the mass up there, we have potential energy mgh. If you drop the ball and let it fall through h, gravity does the work now translating the energy into kinetic energy.

$$mgh = \frac{1}{2}mv^2$$

The potential is the potential energy per unit mass: $V = gh$. We can express this potential as a function of z: $V(z) = gz$. The force of gravity is down, so we write

$$\vec{F} = -mg\,\hat{k}$$

Note that the gravitational field

$$\vec{g} = -g\,\hat{k} = -\frac{d(gz)}{dz}\hat{k} = -\frac{dV(z)}{dz}\hat{k}$$

This is negative the gradient of the potential. In general we can write

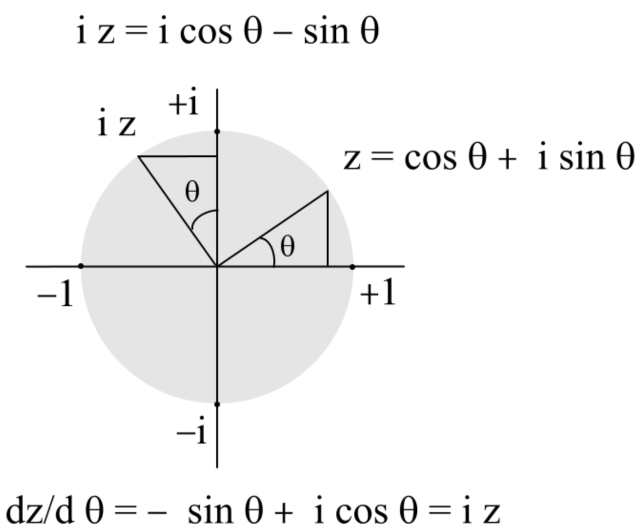
$$\vec{g} = -\nabla V \,.$$

Analogous to this we define the electric potential V such that

$$\vec{E} = -\nabla V \,.$$

The first Maxwell Equation $\nabla \cdot \vec{E} = \dfrac{\rho}{\varepsilon_0}$ leads us to $\nabla^2 V = -\dfrac{\rho}{\varepsilon_0}$. This is called Poisson's equation. If the density is zero, this reduces to Laplace's equation.

2. Euler's Formula (This is Our Jewel).



Physicist Richard Feynman, in his famous lectures on physics delivered at Caltech and available in book form, states that the Euler Formula is the most remarkable formula in mathematics! He states "This is our jewel."

Start with

$$z = \cos\theta + i\sin\theta \,,$$

where $i = \sqrt{-1}$.

PM1 (Practice Problem). Show that multiplying the value of any point on the circumference of the circle in the figure takes you to a point 90° counterclockwise.

Note by reference to the figure that

$$\frac{dz}{d\theta} = iz \,.$$

This leads to $e^{i\theta}$, which gives the "jewel."

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Now comes the secret to connecting i, π and e. It is through our study of the unit circle in the complex plane (above figure). Let θ = π in the Euler Formula $e^{i\theta} = \cos\theta + i\sin\theta$:

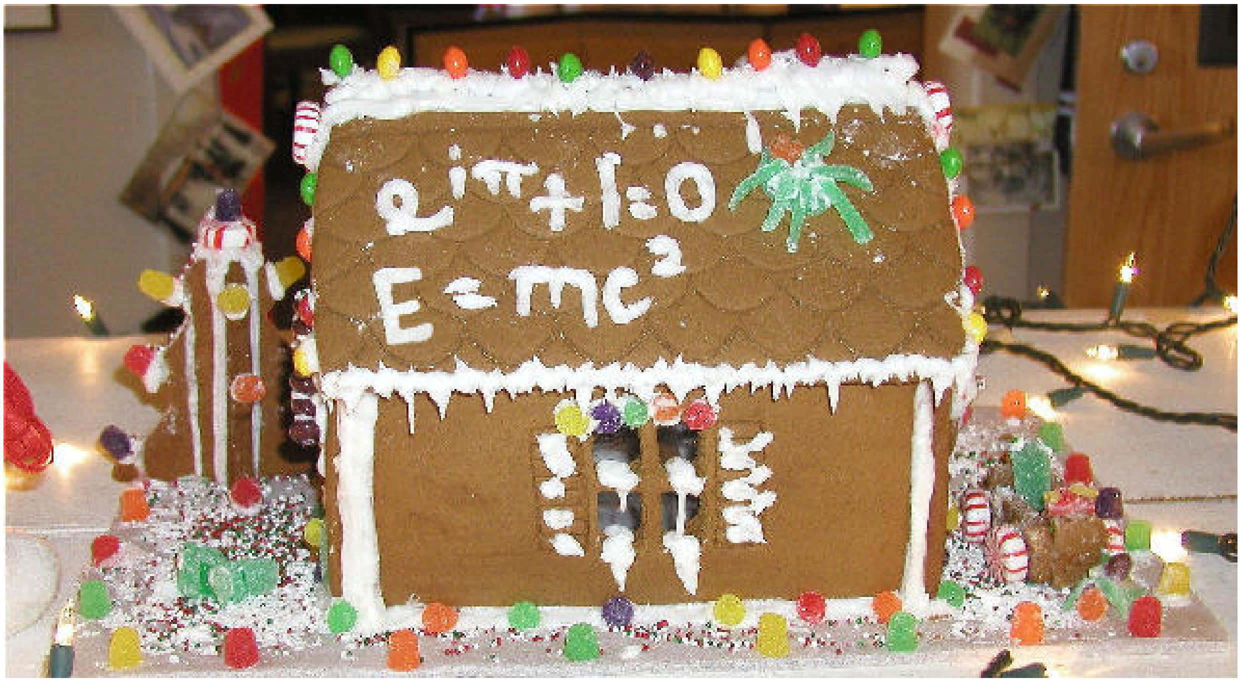
$$e^{i\pi} = -1 \,.$$

This is equivalent to the following magic formula, called *Euler's Identity*.

$$e^{i\pi} + 1 = 0$$

This is our opal.

The Rhoades-Robinson Gingerbread House (2004)



To see the full Gingerbread Album, go to:

http://www.mjtruiz.com/gingerbread.php

You might enjoy reading the explanations I prepared there for the smart high school student.

PM2 (Practice Problem). From

$$e^{i\theta} = \cos\theta + i\sin\theta$$

show that

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \quad \text{and} \quad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$

3. Two Trig Integrals We Will Need. In this section n and m are numbers 1, 2, 3, etc.

$$I_1 = \int_0^{2\pi} \sin^2(n\theta)\,d\theta = \int_0^{2n\pi} \sin^2(z)\frac{dz}{n} = \frac{1}{n}\int_0^{2n\pi} \sin^2(z)\,dz$$

Integrating over a whole number of 2π intervals, it doesn't matter if we are integrating the cosine or sine. **Note: This is true in our case due to the integral multiple of 2π.**

$$I_1 = \frac{1}{2n}\int_0^{2n\pi} \left[\cos^2 z + \sin^2 z\right]dz = \frac{1}{2n}\int_0^{2n\pi} dz = \frac{1}{2n}2n\pi = \pi$$

Now for our second integral consider the following where $n \neq m$.

$$I_2 = \int_0^{2\pi} \sin(n\theta)\sin(m\theta)\,d\theta$$

We use $\sin(n\theta)\sin(m\theta) = \left[\frac{e^{in\theta} - e^{-in\theta}}{2i}\right]\left[\frac{e^{im\theta} - e^{-im\theta}}{2i}\right]$.

Since $n \neq m$, all integrals have the form where p below is a nonzero integer.

$$\int_0^{2\pi} e^{ip\theta}dx = \left.\frac{e^{ip\theta}}{ip}\right|_0^{2\pi} = \frac{1}{ip}\left[\cos(p\theta) + i\sin(p\theta)\right]\Big|_0^{2\pi} = 0$$

The cosine part gives $\cos(2\pi p) - \cos(0) = 1 - 1 = 0$ since p is a whole number and 2p is an even number. This takes us around the circle p times, always giving 1 for the cosine . For the sine, $\sin(2\pi p) = 0$ always. Therefore,

$$\int_0^{2\pi} \sin(n\theta)\sin(m\theta)\,d\theta = \pi\delta_{nm}\,.$$

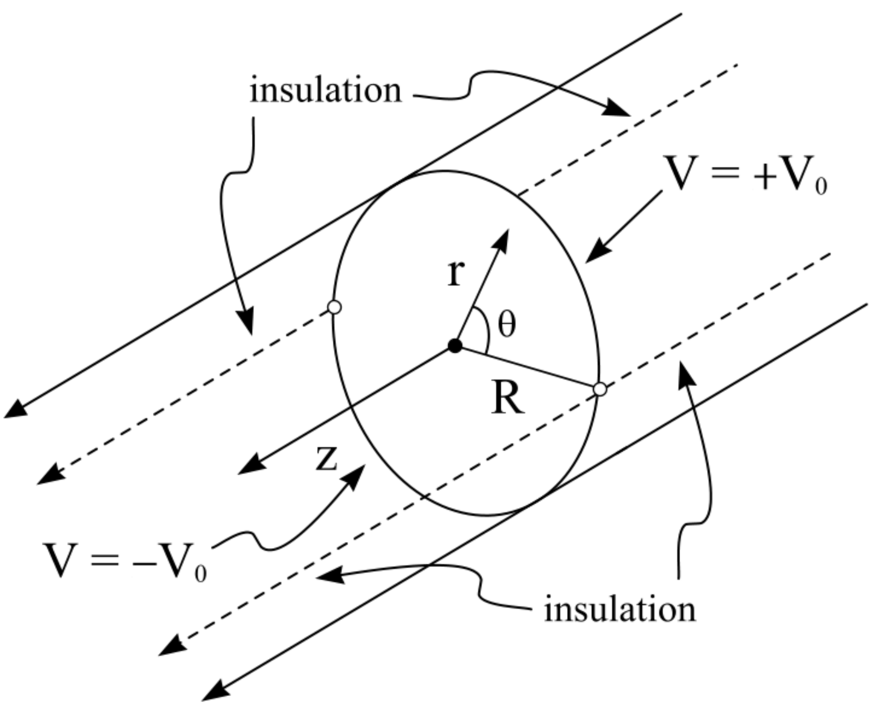
This is referred to as the orthogonality of sine functions.

M2. Laplace's Equation in Cylindrical Coordinates

This problem is one of the richest ones you will ever encounter in your physics courses. There is an amazing amount of mathematical physics here. The source for this marvelous problem comes from the superb text *Mathematical Physics* by Eugene Butkov (Addison-Wesley, Reading, MA, 1968). Here are ingredients in our analysis.

- Laplace's Equation
- Separation of Variables
- Solving a Second-Order Differential Equation
- The Method of Frobenius (Power Series Solution)
- Boundary Conditions (Boundary-Value Problem)
- Orthogonality of Functions
- A Surprise Ending!

Consider a "Long Hollow Cylinder Shell" with Potential $V(R,\theta,z)$ such that on the shell (r = R), $V(R,\theta,z) = V_o$ for $0 < \theta < \pi$ and $V(R,\theta,z) = -V_o$ for $-\pi < \theta < 0$. Imagine a nearby circuit, where we connect a wire at 9 volts to the top part of the cylindrical shell and -9 volts to the bottom. Note the small insulation rods.



We wish to solve Laplace's equation since there is no charge density inside.

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial V}{\partial r}) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} + \frac{\partial^2 V}{\partial z^2} = 0 \,.$$

Since we have cylindrical symmetry, our potential will not be a function of z. Therefore we have $V(r,\theta)$ and the z-derivative is zero. This leaves us with the simpler

$$\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial V}{\partial r}) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} = 0 \,.$$

We use the method called separation of variables. We set our solution to a product of two functions, one for each of the independent variables.

$$V(r,\theta) = f(r)g(\theta)$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial[f(r)g(\theta)]}{\partial r}) + \frac{1}{r^2}\frac{\partial^2[f(r)g(\theta)]}{\partial \theta^2} = 0$$

$$\frac{1}{r}\frac{d}{dr}\left[r\frac{df}{dr}\right]g + \frac{1}{r^2}f\frac{d^2 g}{d\theta^2} = 0$$

The trick now is to divide by fg so we can start to separate "r-stuff" from "theta-stuff."

$$\frac{1}{r}\frac{1}{f}\frac{d}{dr}\left[r\frac{df}{dr}\right] + \frac{1}{r^2}\frac{1}{g}\frac{d^2 g}{d\theta^2} = 0$$

Now clear the r so that each part has only one variable.

$$\frac{r}{f}\frac{d}{dr}\left[r\frac{df}{dr}\right] + \frac{1}{g}\frac{d^2 g}{d\theta^2} = 0$$

Each of these pieces must be a constant since that is the only way to get zero all the time. Remember that r and θ are variables that are independent of each other.

$$\frac{1}{g}\frac{d^2 g}{d\theta^2} = \lambda$$

$$\frac{d^2 g}{d\theta^2} = \lambda g$$

For $\lambda > 0$, the solutions are exponentials: $e^{\lambda\theta}$ and $e^{-\lambda\theta}$. But this can't be since the boundary conditions require that if we move along the angle a full circle we have to come back to where we started. This condition is stated mathematically as

$$g(\theta + 2\pi) = g(\theta) .$$

This boundary condition requires sines and cosines, which we can get if $\lambda < 0$. It is convenient to define $\lambda = -m^2$ so that our differential equation becomes

$$\frac{d^2 g}{d\theta^2} = -m^2 g \quad \text{with solution} \quad g(\theta) = a_m \cos(m\theta) + b_m \sin(m\theta) .$$

The boundary condition $g(\theta + 2\pi) = g(\theta)$ requires $m = 0, 1, 2, 3, \ldots$ The radial equation is

$$\frac{r}{f}\frac{d}{dr}\left[r\frac{df}{dr} \right] + \lambda = 0 \text{, i.e., } \frac{r}{f}\frac{d}{dr}\left[r\frac{df}{dr} \right] - m^2 = 0 .$$

Write this as

$$r\frac{d}{dr}\left[r\frac{df}{dr} \right] - m^2 f = 0 .$$

First consider the m = 0 case. Then,

$$r\frac{d}{dr}\left[r\frac{df}{dr} \right] = 0 \text{, which gives } r\left[\frac{df}{dr} + r\frac{d^2 f}{dr^2} \right] = 0 ,$$

using the product rule for differentiation.

Working with the first derivative $f' = \frac{df}{dr}$, this differential equation becomes

$$r\left[f' + r\frac{df'}{dr} \right] = 0 .$$

Since r is arbitrary, what's inside the brackets must vanish:

$$f' + r\frac{df'}{dr} = 0 .$$

Therefore we have the following.

$$r\frac{df'}{dr} = -f'$$

$$\frac{df'}{f'} = -\frac{dr}{r}$$

Integrating, we find

$$\ln f' = -\ln r + \ln C ,$$

where we write our constant as log of a constant. The above equation can be expressed as

$$\ln f' = \ln\frac{C}{r}$$ and finally as $$f' = \frac{C}{r} .$$

PM3 (Practice Problem). Show that $f = C\ln r + D$, where d is a constant.

This solution is unacceptable since we have a blow-up at r = 0 as the ln(r) runs off to negative infinity. Therefore, there is no physical solution for m = 0.

This leaves us with

$$r\frac{d}{dr}\left[r\frac{df}{dr} \right] - m^2 f = 0$$ and

$$g(\theta) = a_m \cos(m\theta) + b_m \sin(m\theta)$$ where m = 1, 2, 3, ...

Using the product rule of differentiation in the radial equation leads to

$$r\left[\frac{df}{dr}+r\frac{d^2 f}{dr^2}\right]-m^2 f=0 \text{, i.e., } r^2\frac{d^2 f}{dr^2}+r\frac{df}{dr}-m^2 f=0 .$$

This is Euler's differential equation.

$$r^2 f''+rf'-m^2 f=0$$

The classic "Method of Frobenius" is to look for a solution in terms of a power series.

$$f(r)=\sum_{k=1}^{\infty} c_k r^k$$

Plug into your equation f(r), f'(r), and f"(r) .

$$f'(r)=\sum_{k=1}^{\infty} kc_k r^{k-1} \quad \text{and} \quad f''(r)=\sum_{k=2}^{\infty} k(k-1)c_k r^{k-2}$$

Note that one derivative kills the first term (the constant) and two derivatives kill off the first two terms. So we start k at 1 and at 2 in the above equations. But notice that we can start them with k = 0 anyway since the factors give zero for us anyway.

The result is the following.

$$r^2\sum_{k=0}^{\infty} k(k-1)c_k r^{k-2}+r\sum_{k=0}^{\infty} kc_k r^{k-1}-m^2\sum_{k=0}^{\infty} c_k r^k=0$$

We are very lucky and arrive at a very simple result below. This does not happen in general as the powers of r are usually different in the various sums.

$$\sum_{k=0}^{\infty} k(k-1)c_k r^k+\sum_{k=0}^{\infty} kc_k r^k-m^2\sum_{k=0}^{\infty} c_k r^k=0$$

$$\sum_{k=0}^{\infty}\left[k(k-1)+k-m^2\right]c_k r^k=0$$

Since r is arbitrary and can be chosen at will, the part inside the brackets must vanish to make the equation true at all times.

$$k(k-1)+k-m^2=0$$

This leads to $k^2-k+k-m^2=0$, which gives $k^2-m^2=0$, $k^2=m^2$, and

$$k=\pm m.$$

There are only two values for k. This leads us to solutions r^m and $\frac{1}{r^m}$. The second solution blows up at r = 0. So we keep the first one. The general solution can be constructed from the product of the functions

$$f(r)=c_m r^m \quad \text{and} \quad g(\theta)=a_m\cos(m\theta)+b_m\sin(m\theta)$$

We can absorb the c_m constant into a_m and b_m, writing

$$f(r)g(\theta)=r^m\left[a_m\cos(m\theta)+b_m\sin(m\theta)\right]$$

Remember that m = 1, 2, 3, ... Therefore, the general solution is

$$V(r,\theta)=\sum_{m=1}^{\infty}r^m\left[a_m\cos(m\theta)+b_m\sin(m\theta)\right].$$

But we have not exploited all the boundary conditions. As r approaches R, we approach $+V_0$ for angles from 0 to π and $-V_0$ for angles from π to 2π. Therefore

$V(r,-\theta)=-V(r,\theta)$, i.e., the potential is an odd function.

Therefore we only have the sine functions. In other words $a_m=0$.

$$V(r,\theta)=\sum_{m=1}^{\infty}r^m b_m\sin(m\theta)$$

What remains is to find the b_m coefficients in. We do this by focusing our attention on our solution at the shell, i.e., r = R.

$$V(R,\theta) = \sum_{m=1}^{\infty} R^m b_m \sin(m\theta)$$

We multiply both sides by $\sin(nx)$ and integrate.

$$\int_0^{2\pi} \sin(n\theta) V(R,\theta) d\theta = \int_0^{2\pi} \sin(n\theta) \sum_{m=1}^{\infty} R^m b_m \sin(m\theta) d\theta$$

We work on both sides as we go. Since V is an odd function and sine is odd, the product in the integrand on the left side is even. So we can integrate over half the angle range and include a factor of 2. For the right side, our orthogonality relation kicks in.

$$2\int_0^{\pi} \sin(n\theta) V_0 d\theta = \sum_{m=1}^{\infty} R^m b_m \int_0^{2\pi} \sin(n\theta) \sin(m\theta) d\theta$$

$$2V_0 \int_0^{\pi} \sin(n\theta) d\theta = \sum_{m=1}^{\infty} R^m b_m \pi \delta_{nm}$$

$$-2V_0 \left. \frac{\cos(n\theta)}{n} \right|_0^{\pi} = R^n b_n \pi$$

Note that $\cos(n_{odd}\pi) = -1$ and $\cos(n_{even}\pi) = +1$. Therefore, only the odd n cases are nonzero.

We find $-2V_0 \dfrac{(-1-1)}{n} = R^n b_n \pi$, where n is odd.

We want to solve for the b coefficients.

$$\frac{4V_0}{n} = R^n b_n \pi$$

$$b_n = \frac{1}{R^n}\frac{4V_0}{\pi n}$$

Substituting the b coefficients in

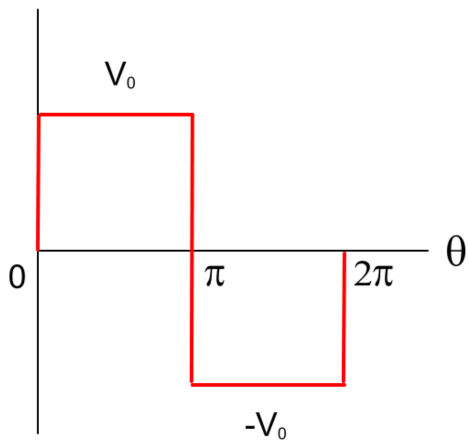
$$V(r,\theta) = \sum_{n=1,3,5,...}^{\infty} r^n b_n \sin(n\theta)$$, we get the following.

$$V(r,\theta) = \sum_{n=1,3,5,...}^{\infty} r^n \frac{1}{R^n}\frac{4V_0}{\pi n}\sin(n\theta)$$

$$V(r,\theta) = \frac{4V_0}{\pi}\sum_{n=1,3,5,...}^{\infty}\left[\frac{r}{R}\right]^n \frac{\sin(n\theta)}{n}$$

And now we reach the surprise. The limiting case r = R gives us

V on the Shell (at r = R)



$$V(R,\theta) = \frac{4V_0}{\pi}\sum_{n=1,3,5,...}^{\infty}\frac{\sin(n\theta)}{n}$$

What's this?

It's the Fourier Series for a Square Wave!

Check out the graph of our potential on the cylindrical shell (see left figure). It's a square wave.

$$V(R,\theta) = \frac{4V_0}{\pi}\left[\sin\theta + \frac{\sin(3\theta)}{3} + \frac{\sin(5\theta)}{5} + ...\right]$$

With sleight of hand, our solution leads to the Fourier series of a square wave as a byproduct. This is such a rich problem!

PM4 (Practice). Take the gradient to find the electric field: $\vec{E} = -\nabla V(r,\theta)$.