Electromagnetic Theory
Prof. Ruiz, UNC Asheville, doctorphys on YouTube
Chapter N Notes. Poisson's Equation

N1. Review of Cylindrical Coordinates

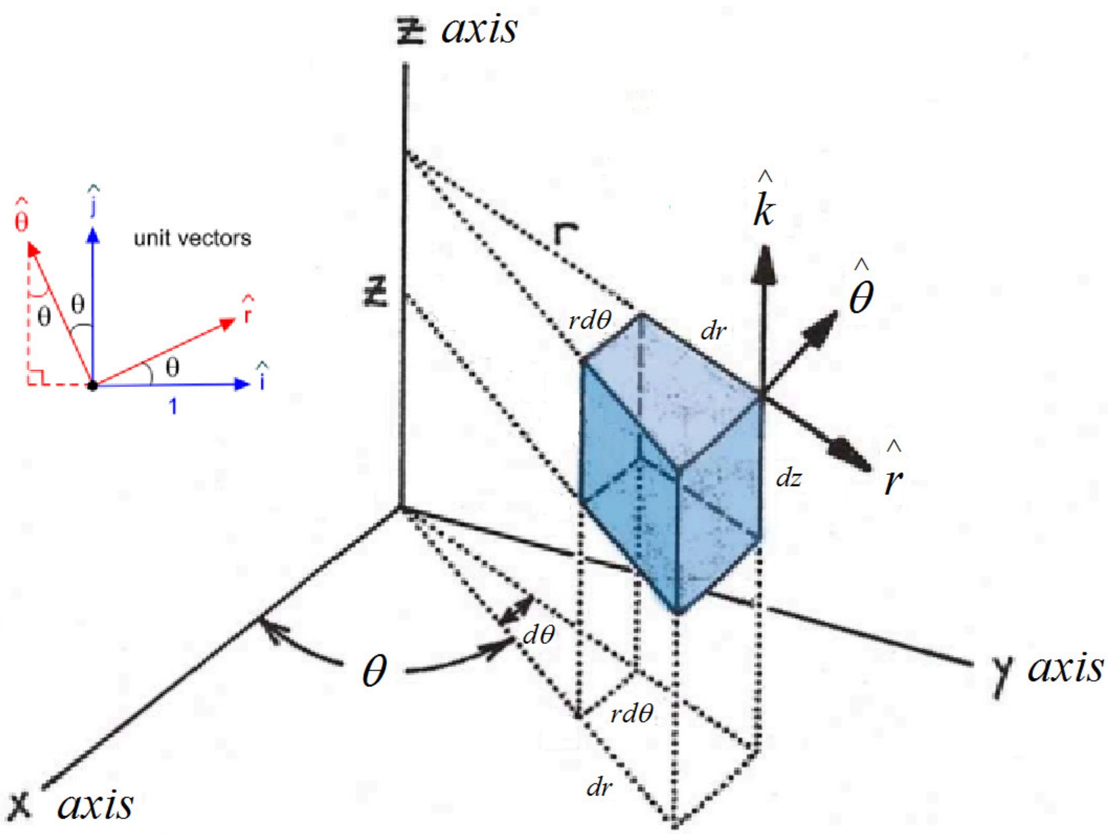


Figure Adapted from Professor Kurt E. Oughstun, School of Engineering
College of Engineering & Mathematical Sciences, University of Vermont

From before we have the following with the notation given above.

$$x = r\cos\theta \qquad y = r\sin\theta \qquad r = \sqrt{x^2 + y^2} \qquad \tan\theta = \frac{y}{x}$$

$$\hat{r} = \cos\theta\,\hat{i} + \sin\theta\,\hat{j} \qquad \hat{\theta} = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j} \qquad \hat{k} = \hat{k}$$

Professor Oughstun derives the gradient in cylindrical coordinates from scratch in his EE 141 Electromagnetic Field Theory I course. This is nice to see as it gives us confidence that our general result from curvilinear coordinates is correct. We will do this now with the additional feature that we will NOT look up any derivatives.

This is the kind of workout I was referring to when I gave you the Laplacian derivation for a summer project. Watch below for the workout. Before doing the Laplacian you might do the following gradient derivation in spherical coordinates.

$$x = r\cos\theta \qquad y = r\sin\theta \qquad r = \sqrt{x^2 + y^2} \qquad \tan\theta = \frac{y}{x}$$

$$\hat{r} = \cos\theta\,\hat{i} + \sin\theta\,\hat{j} \qquad \hat{\theta} = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j} \qquad \hat{k} = \hat{k}$$

We want to check $\nabla f = \frac{\partial f}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\theta} + \frac{\partial f}{\partial z}\hat{k}$.

Start with $\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$ and substitute things.

$$\nabla f = \left[\frac{\partial f}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial x}\right]\hat{i} + \left[\frac{\partial f}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial y}\right]\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

PN1 (Practice Problem). Show $\hat{i} = \cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}$ and $\hat{j} = \sin\theta\,\hat{r} + \cos\theta\,\hat{\theta}$.

PN2 (Practice Problem). Show $\frac{\partial r}{\partial x} = \frac{x}{r} = \cos\theta$ and $\frac{\partial r}{\partial y} = \frac{y}{r} = \sin\theta$.

Note that $\theta = \tan^{-1}\frac{y}{x} \equiv \tan^{-1} u$. Instead, consider $\frac{\partial \theta}{\partial x} = \frac{d\theta}{d\tan\theta}\frac{\partial \tan\theta}{\partial x}$.

$$\frac{d\tan\theta}{d\theta} = \frac{d}{d\theta}\left[\sin\theta\cos^{-1}\theta\right] = \cos\theta\cos^{-1}\theta + \sin\theta\frac{(-1)(-\sin\theta)}{\cos^2\theta} = 1 + \tan^2\theta$$

Now we consider the "flip" derivative: $\frac{d\theta}{d\tan\theta} = \frac{1}{1+\tan^2\theta} = \frac{1}{1+(y/x)^2}$.

$$\frac{\partial \theta}{\partial x} = \frac{1}{1+(y/x)^2}\frac{\partial \tan\theta}{\partial x} = \frac{1}{1+(y/x)^2}\left[-\frac{y}{x^2}\right] \text{ from } \tan\theta = \frac{y}{x}$$

$$\frac{\partial\theta}{\partial x}=\frac{1}{1+(y/x)^2}\left[-\frac{y}{x^2}\right]=\frac{-y}{x^2+y^2}=-\frac{r\sin\theta}{r^2}=-\frac{1}{r}\sin\theta$$

PN3 (Practice Problem). Show that $\frac{\partial\theta}{\partial y}=\frac{1}{r}\cos\theta$.

Summary of what we have so far.

$$\nabla f=\left[\frac{\partial f}{\partial r}\frac{\partial r}{\partial x}+\frac{\partial f}{\partial\theta}\frac{\partial\theta}{\partial x}\right]\hat{i}+\left[\frac{\partial f}{\partial r}\frac{\partial r}{\partial y}+\frac{\partial f}{\partial\theta}\frac{\partial\theta}{\partial y}\right]\hat{j}+\frac{\partial f}{\partial z}\hat{k}$$

$$\hat{i}=\cos\theta\,\hat{r}-\sin\theta\,\hat{\theta}\qquad \hat{j}=\sin\theta\,\hat{r}+\cos\theta\,\hat{\theta}$$

$$\frac{\partial r}{\partial x}=\frac{x}{r}=\cos\theta\qquad \frac{\partial r}{\partial y}=\frac{y}{r}=\sin\theta\qquad \frac{\partial\theta}{\partial x}=-\frac{1}{r}\sin\theta\qquad \frac{\partial\theta}{\partial y}=\frac{1}{r}\cos\theta$$

$$\nabla f=\left[\frac{\partial f}{\partial r}\cos\theta+\frac{\partial f}{\partial\theta}(-\frac{1}{r}\sin\theta)\right](\cos\theta\,\hat{r}-\sin\theta\,\hat{\theta})$$

$$+\left[\frac{\partial f}{\partial r}\sin\theta+\frac{\partial f}{\partial\theta}(\frac{1}{r}\cos\theta)\right](\sin\theta\,\hat{r}+\cos\theta\,\hat{\theta})+\frac{\partial f}{\partial z}\hat{k}$$

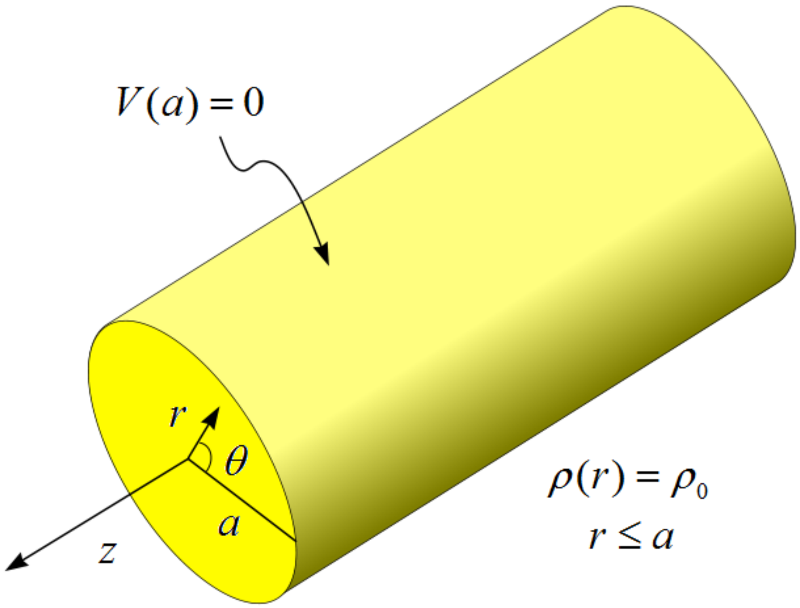
PN4 (Practice Problem). Show that the above equation gives the expected result shown below.

$$\nabla f=\frac{\partial f}{\partial r}\hat{r}+\frac{1}{r}\frac{\partial f}{\partial\theta}\hat{\theta}+\frac{\partial f}{\partial z}\hat{k}$$

This is what we get using the curvilinear formula with the proper scale factors.

$$\nabla f=\frac{1}{h_1}\frac{\partial f}{\partial q_1}\hat{e}_1+\frac{1}{h_2}\frac{\partial f}{\partial q_2}\hat{e}_2+\frac{1}{h_3}\frac{\partial f}{\partial q_3}\hat{e}_3$$

N2. Cylinder with Constant Charge. He we do the infinitely long cylinder of radius a with constant charge using Gauss's Law from Intro Physics. Then in the next section we do it the long way using Poisson's equation from our course. Gauss's Law is



$$\oiint \vec{E} \cdot \vec{dA} = \frac{Q_{in}}{\varepsilon_0} .$$

The charge density is

$$\rho(r) = \rho_0$$

for $r \le a$ and there is no charge for $r > a$.

Remember that the cylinder can be taken to be infinitely long.

Gauss's Law for $r < a$: $E_{in}(2\pi r l) = \dfrac{\rho_0 \pi r^2 l}{\varepsilon_0}$ using a Gaussian cylindrical surface defined by a radius r and length l . We get $E_{in}(r) = \dfrac{\rho_0 r}{2\varepsilon_0}$ where "in" means inside.

For $r > a$, $E_{out}(2\pi r l) = \dfrac{\rho_0 \pi a^2 l}{\varepsilon_0}$, giving for outside: $E_{out}(r) = \dfrac{\rho_0 a^2}{2\varepsilon_0} \dfrac{1}{r}$.

Note that $E_{in}(a) = \dfrac{\rho_0 a}{2\varepsilon_0}$ equals $E_{out}(a) = \dfrac{\rho_0 a^2}{2\varepsilon_0} \dfrac{1}{a} = \dfrac{\rho_0 a}{2\varepsilon_0}$.

The potential is related to the electric field as $E(r) = -\dfrac{dV(r)}{dr}$ since the electric field is radial. Then, $V(r) = -\int E(r)dr + const$.

$$V_{in}(r) = -\int \frac{\rho_0 r}{2\varepsilon_0} dr = -\frac{\rho_0 r^2}{4\varepsilon_0} + A$$

To match the boundary condition $V(a)=0$ we need $-\frac{\rho_0 a^2}{4\varepsilon_0}+A=0$, which gives $A=\frac{\rho_0 a^2}{4\varepsilon_0}$. Then the potential function on the inside will match at the boundary.

Our result is $V_{in}(r)=\frac{\rho_0 a^2}{4\varepsilon_0}-\frac{\rho_0 r^2}{4\varepsilon_0}$, i.e., $V_{in}(r)=\frac{\rho_0 a^2}{4\varepsilon_0}\left[1-\frac{r^2}{a^2}\right]$.

Note that we can call the zero potential reference anywhere we like just as we do in mechanics with mgh for gravitational potential energy near the Earth. You can take h = 0 to be the ground or the top of the table. Here we choose the surface of the cylinder to have potential zero.

For the outside, $E_{out}(r)=\frac{\rho_0 a^2}{2\varepsilon_0}\frac{1}{r}$ and $V(r)=-\frac{\rho_0 a^2}{2\varepsilon_0}\int\frac{1}{r}dr+const$.

$$V_{out}(r)=-\frac{\rho_0 a^2}{2\varepsilon_0}\ln r+B$$

To match at the boundary, $V_{out}(a)=-\frac{\rho_0 a^2}{2\varepsilon_0}\ln a+B=0$, which gives $B=\frac{\rho_0 a^2}{2\varepsilon_0}\ln a$. The potential on the outside is then

$$V_{out}(r)=-\frac{\rho_0 a^2}{2\varepsilon_0}\left[\ln r-\ln a\right]=-\frac{\rho_0 a^2}{2\varepsilon_0}\ln\frac{r}{a}.$$

Nice to see the dimensionless argument in that logarithm. A summary is below.

$$\vec{E}_{in}(r)=\frac{\rho_0 r}{2\varepsilon_0}\hat{r} \qquad \vec{E}_{out}(r)=\frac{\rho_0 a^2}{2\varepsilon_0}\frac{1}{r}\hat{r}$$

$$V_{in}(r)=\frac{\rho_0 a^2}{4\varepsilon_0}\left[1-\frac{r^2}{a^2}\right] \qquad V_{out}(r)=-\frac{\rho_0 a^2}{2\varepsilon_0}\ln\frac{r}{a}$$

Now we proceed to the upper-level version, using techniques from our course to get the same answers we have just found using methods from introductory physics.

N3. Poisson's and Laplace's Equations. The upper-level definition of the potential involves the gradient.

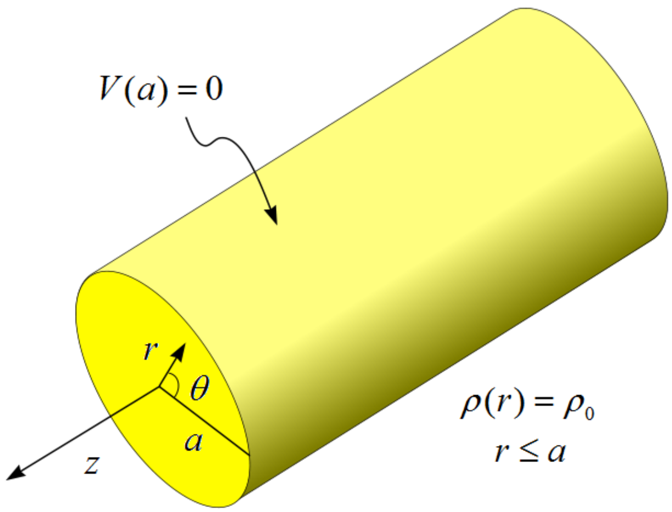
$$\vec{E} = -\nabla V$$

The first Maxwell equation $\nabla \cdot \vec{E} = \dfrac{\rho}{\varepsilon_0}$ leads to Poisson's equation,

$$\nabla^2 V = -\frac{\rho}{\varepsilon_0}\text{, since } \nabla \cdot \vec{E} = -\nabla \cdot \nabla V = -\nabla^2 V\ .$$

In cylindrical coordinates, Poisson's equation is

$$\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial V}{\partial r}) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} + \frac{\partial^2 V}{\partial z^2} = -\frac{\rho}{\varepsilon_0}$$



We now do the same problem the longer way to get practice with solving Poisson's equation.

Outside the cylinder the charge density is zero and Poisson's equation reduces to Laplace's equation.

On the inside, the charge density function is a constant:

$$\rho(r) = \rho_0\ .$$

This means very nice symmetry and much simplification. The potential is only a function of r.

$$V = V(r, \theta, z) = V(r)$$

Therefore, Poisson's equation will not have the angle and z derivatives since they result in zero. We can also replace partial derivatives with the standard derivatives as our problem reduces to just the r variable.

$$\frac{1}{r}\frac{d}{dr}(r\frac{dV}{dr}) = -\frac{\rho_0}{\varepsilon_0}$$

Inside - Solving Poisson's Equation: $\frac{1}{r}\frac{d}{dr}(r\frac{dV}{dr}) = -\frac{\rho_0}{\varepsilon_0}$

$$\frac{d}{dr}(r\frac{dV}{dr}) = -\frac{\rho_0}{\varepsilon_0}r$$

$$r\frac{dV}{dr} = -\frac{\rho_0}{\varepsilon_0}\frac{r^2}{2} + A$$, where A is a constant.

$$\frac{dV}{dr} = -\frac{\rho_0}{2\varepsilon_0}r + \frac{A}{r}$$

We integrate again introducing a second constant B.

$$V_{in}(r) = -\frac{\rho_0}{4\varepsilon_0}r^2 + A\ln r + B$$

Note there are two solutions for a second-order differential equation. We have the quadratic function and the logarithm. But the logarithm shoots off to minus infinity when r = 0. Therefore, A must be 0. This analysis involves looking at a boundary condition. Though we do not know the exact value of the potential at r = 0, we do know it can't shoot off to infinity. This leaves

$$V_{in}(r) = -\frac{\rho_0}{4\varepsilon_0}r^2 + B$$

We determine B from the boundary condition at r = a.

$$V_{in}(a) = -\frac{\rho_0}{4\varepsilon_0}a^2 + B = 0 \quad \text{and} \quad B = \frac{\rho_0}{4\varepsilon_0}a^2$$

$$V_{in}(r) = \frac{\rho_0 a^2}{4\varepsilon_0}\left[1 - \frac{r^2}{a^2}\right]$$

Outside - Solving Laplace's Equation: $\frac{1}{r}\frac{d}{dr}(r\frac{dV}{dr}) = 0$

$$\frac{d}{dr}(r\frac{dV}{dr}) = 0$$

$r\frac{dV}{dr} = C$, where C is a constant.

$$\frac{dV}{dr} = \frac{C}{r}$$

We integrate introducing a second constant D.

$$V_{out}(r) = C\ln r + D$$

Here the logarithm does not get discarded since r is never zero for the outside region.

We determine D from the boundary conditions. Watch how when we do this that logarithm of r will become a logarithm of a dimensionless quantity. We know we can't take a log of a distance in meters. We take logs of pure numbers. If we do our physics right, the math will come out correctly.

$$V_{out}(r) = C\ln r + D$$

$$V_{out}(a) = C\ln a + D = 0$$

$$D = -C\ln a$$

$$V_{out}(r) = C\ln r - C\ln a$$

$$V_{out}(r) = C\ln\frac{r}{a}$$

We arrive at the natural logarithm of a dimensionless quantity as expected.

Summary:

$$V_{in}(r) = \frac{\rho_0 a^2}{4\varepsilon_0}\left[1 - \frac{r^2}{a^2}\right] \qquad V_{out}(r) = C\ln\frac{r}{a}$$

But what about C? We find C by matching the electric fields at the boundary. Matching electric fields means we match the derivatives of the potential at the boundary.

$$\left.\frac{dV_{in}(r)}{dr}\right|_{r=a} = \left.\frac{dV_{out}(r)}{dr}\right|_{r=a}$$

With $u = \frac{r}{a}$, note that $\frac{d\ln(r/a)}{dr} = \frac{d\ln u}{du}\frac{du}{dr} = \frac{1}{u}\frac{du}{dr} = \frac{a}{r}\frac{1}{a} = \frac{1}{r}$.

Using this result for the derivative on the outside, we obtain

$$\frac{\rho_0 a^2}{4\varepsilon_0}\left[-\frac{2r}{a^2}\right]_{r=a} = C\left.\frac{1}{r}\right|_{r=a}$$

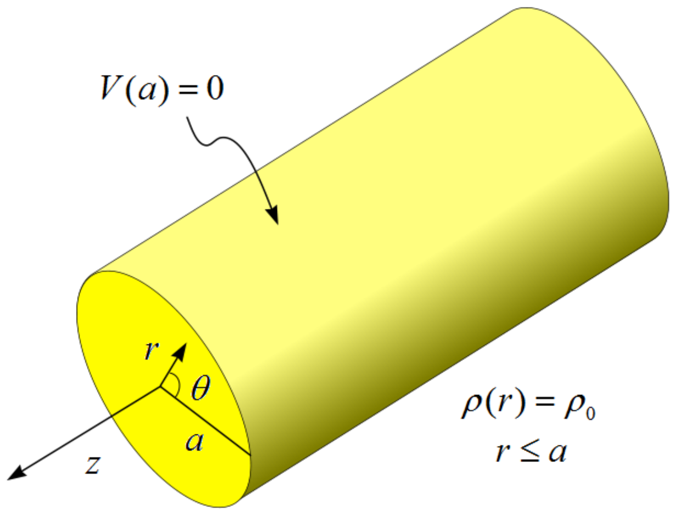
$$\frac{\rho_0 a^2}{4\varepsilon_0}\left[-\frac{2a}{a^2}\right] = C\left.\frac{1}{a}\right|$$

$$\frac{\rho_0 a^2}{4\varepsilon_0}\left[-\frac{2a^2}{a^2}\right] = C \quad \text{and} \quad C = -\frac{\rho_0 a^2}{2\varepsilon_0}$$

Substituting in for the constant in $V_{out}(r) = C\ln\frac{r}{a}$ leads to the second equation below. The first equation is our result for the inside which we arrived at earlier.

$$V_{in}(r) = \frac{\rho_0 a^2}{4\varepsilon_0}\left[1 - \frac{r^2}{a^2}\right] \quad \text{and} \quad V_{out}(r) = -\frac{\rho_0 a^2}{2\varepsilon_0}\ln\frac{r}{a}.$$

Let's double check that the electric fields come out right taking the gradients.



We have our potentials.

$$V_{in}(r) = \frac{\rho_0 a^2}{4\varepsilon_0}\left[1 - \frac{r^2}{a^2}\right]$$

$$V_{out}(r) = -\frac{\rho_0 a^2}{2\varepsilon_0}\ln\frac{r}{a}$$

The electric field is found by the negative gradient $\vec{E} = -\nabla V$.

We arrived at the gradient in cylindrical coordinates from our general formulas in curvilinear coordinates and calculating this one out in detail in our first section of this chapter. The result is

$$\nabla f = \frac{\partial f}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\theta} + \frac{\partial f}{\partial z}\hat{k} .$$

For our case there is no dependence on the angle or z-direction. So we have the simpler version with a regular derivative as there is only one variable, namely r.

$$\nabla V(r) = \frac{dV(r)}{dr}\hat{r}$$

PN5 (Practice Problem). Use $\vec{E} = -\nabla V$ to show that our potentials

$$V_{in}(r) = \frac{\rho_0 a^2}{4\varepsilon_0}\left[1 - \frac{r^2}{a^2}\right] \quad \text{and} \quad V_{out}(r) = -\frac{\rho_0 a^2}{2\varepsilon_0}\ln\frac{r}{a}$$

give us what we already know, namely,

$$\vec{E}_{in}(r) = \frac{\rho_0 r}{2\varepsilon_0}\hat{r} \quad \text{and} \quad \vec{E}_{out}(r) = \frac{\rho_0 a^2}{2\varepsilon_0}\frac{1}{r}\hat{r} .$$