Electromagnetic Theory
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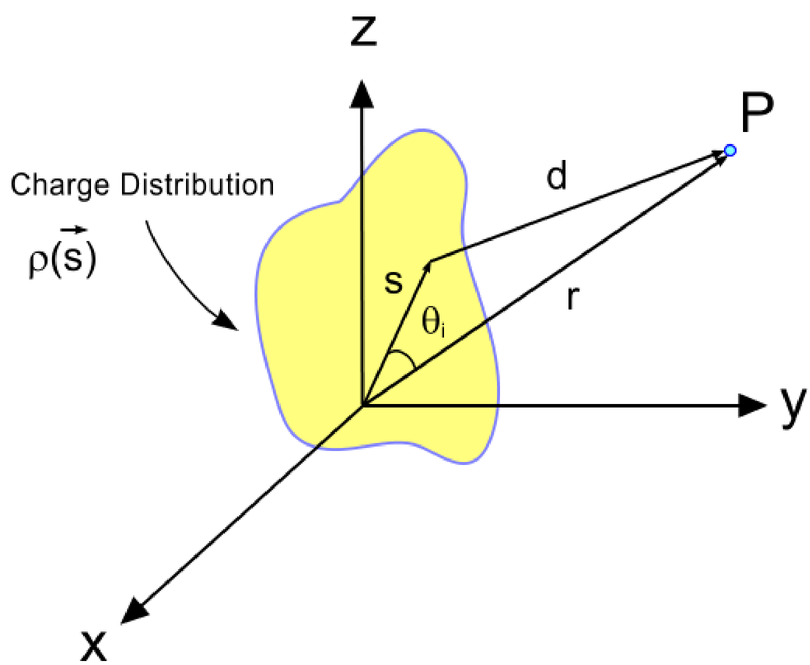
Chapter O Notes. Multipole Expansion

O1. Potential for Arbitrary Charge Distribution

A charge distribution designated by the yellow region is located near the origin. The charge density is $\rho = \rho(\vec{s})$, where we need to integrate over the charge region the source $\vec{s}$ vector roams over the charge area. You can consider our charge density as shorthand for $\rho = \rho(s_x, s_y, s_z)$ and think of "s" as standing for source of charge.

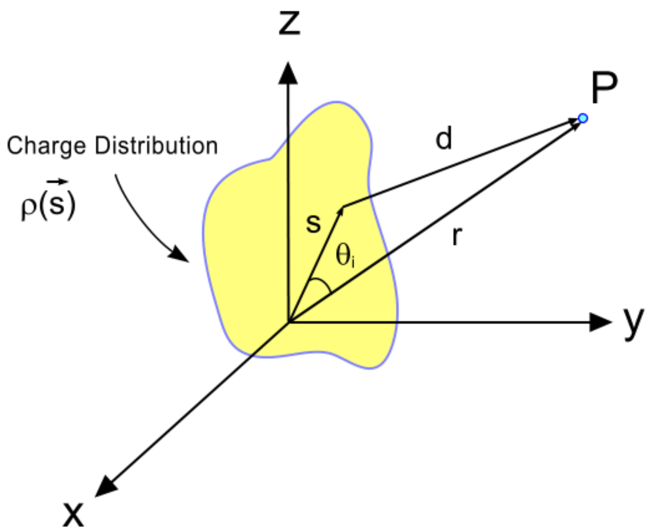


To find the total charge is then

$$Q = \int \rho(\vec{s}) d^3\vec{s} \equiv \iiint \rho(s_x, s_y, s_z) ds_x ds_y ds_z$$, which is also

$$Q = \iiint \rho(s, \theta_s, \phi_s) s^2 \sin\theta_s ds d\theta_s d\phi_s \equiv \int_\tau \rho(\vec{s})\, d^3\vec{s} .$$

The tau τ stands for volume and we write one integral sign for compact notation. The three-dimensional volume element is conveniently written as $d^3\vec{s}$.



The potential at point P is

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_\tau \frac{\rho(\vec{s})}{d} d^3\vec{s} \; ,$$

where we sum the effects of each infinitesimal source at the distance d away. The s-vector plus the d-vector gets you to point P from the origin or you can go directly from the origin to P along the r-vector.

$$\vec{r} = \vec{s} + \vec{d}$$

The vector $\vec{d} = \vec{r} - \vec{s}$ and we can also write $V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_\tau \frac{\rho(\vec{s})}{|\vec{r} - \vec{s}|} d^3\vec{s}$.

The notation $|\vec{d}|$ means the magnitude of $\vec{d}$.

Take the dot product between $\vec{d} = \vec{r} - \vec{s}$ and itself.

$$\vec{d} \cdot \vec{d} = (\vec{r} - \vec{s}) \cdot (\vec{r} - \vec{s})$$

$$\vec{d} \cdot \vec{d} = \vec{r} \cdot \vec{r} - \vec{r} \cdot \vec{s} - \vec{s} \cdot \vec{r} + \vec{s} \cdot \vec{s}$$

$$\vec{d} \cdot \vec{d} = \vec{r} \cdot \vec{r} - 2\vec{r} \cdot \vec{s} + \vec{s} \cdot \vec{s}$$

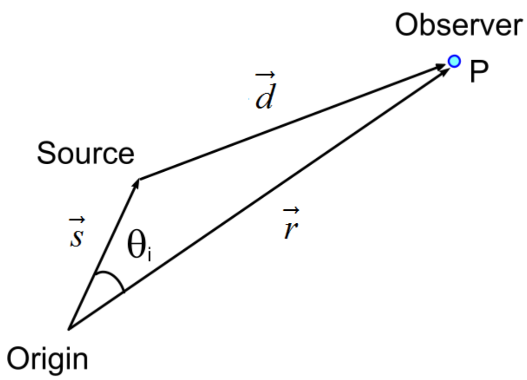
$$d^2 = r^2 - 2rs\cos\theta_i + s^2$$

The angle θ_i is the inner angle between the two vectors $\vec{r}$ and $\vec{s}$. Our last equation is the law of cosines. We can finally write the potential function as

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_\tau \frac{\rho(\vec{s})}{\sqrt{r^2 - 2rs\cos\theta_i + s^2}} d^3\vec{s} \; .$$

O2. Multipole Expansion

Since r is a constant in $V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_\tau \frac{\rho(\vec{s})}{\sqrt{r^2 - 2rs\cos\theta_i + s^2}} d^3\vec{s}$ for our fixed observer, we can factor out a 1/r from our integral. This leaves some r presence in the integral but these appear in the denominator suggesting an expansion if $r >> s$.



$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{1}{r}\int_\tau \frac{\rho(\vec{s})\, d^3\vec{s}}{\sqrt{1 - 2\frac{s}{r}\cos\theta_i + \frac{s^2}{r^2}}}$$

Let $x = \cos\theta_i$ and $h = \frac{s}{r}$. This x-variable is not our usual spatial variable but a special one that is often used to stand for the cosine of an angle.

The denominator is then $\sqrt{1 - 2hx + h^2}$. For the condition $r >> s$ we have $h << 1$. Since $-1 \le x \le 1$, x being a cosine function, then $xh << 1$ also. Therefore, $\varepsilon \equiv -2hx + h^2 << 1$ and we can expand $\frac{1}{\sqrt{1+\varepsilon}}$.

$$\frac{1}{\sqrt{1+\varepsilon}} = (1+\varepsilon)^{-\frac{1}{2}}$$

I prefer to put h after x in ε, i.e., $\varepsilon = -2xh + h^2$.

$$(1+\varepsilon)^{-\frac{1}{2}} = 1 - \frac{1}{2}\varepsilon - \frac{1}{2}(-\frac{3}{2})\frac{\varepsilon^2}{2!} - \frac{1}{2}(-\frac{3}{2})(-\frac{5}{2})\frac{\varepsilon^3}{3!}\cdots$$ with $\varepsilon = -2xh + h^2$,

$$(1+\varepsilon)^{-\frac{1}{2}} = 1 - \frac{1}{2}\varepsilon + \frac{3}{8}\varepsilon^2 - \frac{5}{16}\varepsilon^3 \ldots$$

$$(1+\varepsilon)^{-\frac{1}{2}} = 1 - \frac{1}{2}(-2xh + h^2) + \frac{3}{8}(-2xh + h^2)^2 - \frac{5}{16}(-2xh + h^2)^3 \ldots$$

Now use $(a+b)^2 = a^2 + 2ab + b^2$ and $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$ in

$$(1+\varepsilon)^{-\frac{1}{2}} = 1 - \frac{1}{2}(-2xh + h^2) + \frac{3}{8}(-2xh + h^2)^2 - \frac{5}{16}(-2xh + h^2)^3 \ldots$$

to obtain $(1 - 2hx + h^2)^{-\frac{1}{2}} = 1 + \frac{1}{2}(2xh - h^2) + \frac{3}{8}(4x^2h^2 - 4xh^3 + h^4)$

$$-\frac{5}{16}(-8x^3h^3 + 12x^2h^4 - 6xh^5 + h^6)\ldots .$$

We wish to collect powers of h.

$$(1 - 2hx + h^2)^{-\frac{1}{2}} = 1 + xh + (\frac{3}{2}x^2 - \frac{1}{2})h^2 + (\frac{5}{2}x^3 - \frac{3}{2}x)h^3 + \ldots$$

The factors in front of the powers of h are the Legendre polynomials.

$$(1 - 2hx + h^2)^{-\frac{1}{2}} = P_0(x) + P_1(x)h + P_2(x)h^2 + P_3(x)h^3 + \ldots = \sum_{l=1}^{\infty} P_l(x)h^l$$

The component on the left side of the above equation is called a generating function.

$$G(x,h) = (1 - 2hx + h^2)^{-\frac{1}{2}} = \sum_{l=0}^{\infty} P_l(x)h^l$$

A generating function must have two parameters, one with the independent variable x and the other being a separating variable so that the x-pieces do not get mixed up. The h powers keep the polynomials in x apart from each other.

The first few Legendre polynomials are given below.

$$P_0(x) = 1 \qquad P_1(x) = x \qquad P_2(x) = \frac{1}{2}(3x^2 - 1) \qquad P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

PM1 (Practice Problem). Find $P_4(x)$ and $P_5(x)$.

Substituting $x = \cos\theta_i$ and $h = \frac{s}{r}$, back into

$$(1-2hx+h^2)^{-\frac{1}{2}} = 1 + xh + (\frac{3}{2}x^2 - \frac{1}{2})h^2 + (\frac{5}{2}x^3 - \frac{3}{2}x)h^3 + ...$$

we arrive at

$$(1-2\frac{s}{r}\cos\theta_i + \frac{s^2}{r^2})^{-\frac{1}{2}} = 1 + \cos\theta_i \frac{s}{r} + \frac{1}{2}(3\cos^2\theta_i - 1)\left[\frac{s}{r}\right]^2$$

$$+\frac{1}{2}(5\cos^3\theta_i - 3\cos\theta_i)\left[\frac{s}{r}\right]^3 + ...$$

Finally putting this into $V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{1}{r}\int_\tau \frac{\rho(\vec{s})\,d^3\vec{s}}{\sqrt{1-2\frac{s}{r}\cos\theta_i + \frac{s^2}{r^2}}}$, gives us

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{1}{r}\int_\tau \rho(\vec{s})\,d^3\vec{s} + \frac{1}{4\pi\varepsilon_0}\frac{1}{r^2}\int_\tau \cos\theta_i s\rho(\vec{s})\,d^3\vec{s}$$

$$+\frac{1}{4\pi\varepsilon_0}\frac{1}{r^3}\int_\tau \frac{1}{2}(3\cos^2\theta_i - 1)s^2\rho(\vec{s})\,d^3\vec{s} + ...$$

Each term is called a pole. The first three are known as the monopole, dipole, and quadrupole. You can see why the first is called the monopole - it is the result for one point charge.

Monopole: $V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{1}{r}\int_\tau \rho(\vec{s})\,d^3\vec{s} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r}$

Dipole: $V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^2}\int_\tau \cos\theta_i s\rho(\vec{s})\,d^3\vec{s}$

Quadrupole: $$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^3}\int_\tau \frac{1}{2}(3\cos^2\theta_i - 1)s^2\rho(\vec{s})\,d^3\vec{s}$$

In summation form, we have all the poles in

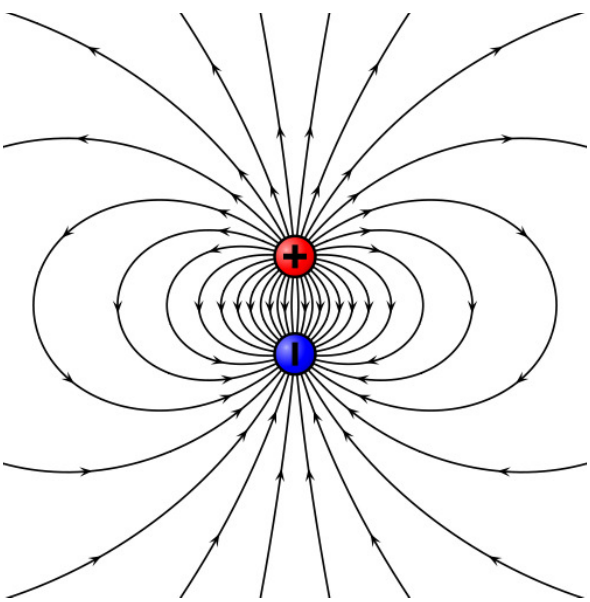
$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\sum_{l=0}^{\infty}\frac{1}{r^{l+1}}\int_\tau P_l(\cos\theta_i)s^l\rho(\vec{s})\,d^3\vec{s}\,.$$

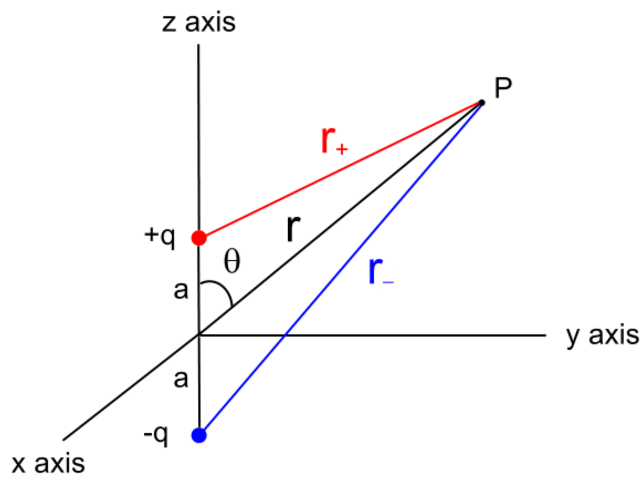
This is our multipole expansion for the potential.

We will investigate these more fully in our chapter on potentials.

PM2 (Practice Problem). Show that the electric potential for a dipole (see figure) is given by $V_{dipole}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{\vec{p}\cdot\hat{r}}{r^2}$, where the dipole moment $\vec{p}$ is defined as $\vec{p} = 2aq\,\hat{k}$. The magnitude of the dipole moment is given by the product of the positive charge and the distance between the positive and negative charges. The direction points from the minus charge to the positive one.

Field lines Courtesy Geek3, Wikipedia.





Then find E_z from $-\frac{\partial V}{\partial z}$ and show that far away up the z-axis your result has an inverse-cube law for the electric field.

END OF CLASS PROPER (NEXT SECTION IS BONUS SECTION you can skim.)

O3. The "Magical" Approach to the Legendre Differential Equation

Adrien-Marie Legendre (1752-1833)
From Wikipedia: 1820 watercolor caricature of Adrien-Marie Legendre by French artist Julien-Leopold Boilly. It's the only existing portrait.

The Legendre differential equation is

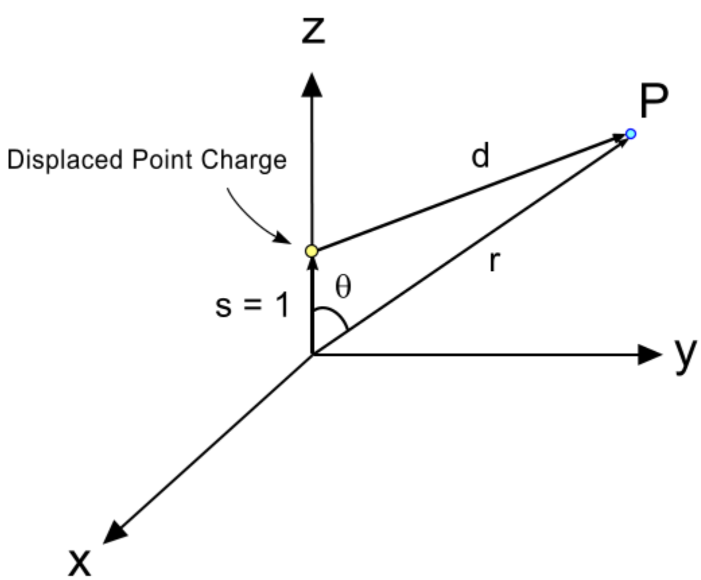
$$(1-x^2)y''-2xy'+l(l+1)y=0\,,$$

where $l=0,1,2,3,\ldots$

The solutions are the Legendre polynomials

$$P_l(x)\,.$$

We have already seen the Legendre polynomials. Here is a magical approach to arrive at how the Legendre polynomials involving the spherical coordinate θ appear in problems with symmetry about the z-axis. I found it in the excellent text *Mathematical Physics* by Eugene Butkov (Addison-Wesley, Reading, MA, 1968), which we have referenced before. The author there considers a point charge displaced a unit distance so z = s = 1. We will study points near the origin this time instead of far away.



The figure at the left shows a point charge, which will we take to have charge Q, displaced from its usual position at the origin to a distance 1 along the z-axis.

We know the potential of a point charge at the origin. It is

$$V(r)=\frac{1}{4\pi\varepsilon_0}\frac{Q}{|\vec{r}|}=\frac{1}{4\pi\varepsilon_0}\frac{Q}{r}\,.$$

For the displaced point we have

$$V(r,\theta)=\frac{1}{4\pi\varepsilon_0}Q=\frac{1}{4\pi\varepsilon_0}\frac{Q}{d}=\frac{Q}{4\pi\varepsilon_0}\frac{1}{\sqrt{1-2r\cos\theta+r^2}}\,.$$

Note that our inner angle θ_i now is our spherical-coordinate angle θ.

The trick now is search for a solution near the origin, i.e., where $r << 1$. But we are expert now in expanding something of the form $\frac{1}{\sqrt{1-2r\cos\theta + r^2}}$ where $\varepsilon = -2r\cos\theta + r^2$ is small. We know

$$G(x,h) = (1-2hx+h^2)^{-\frac{1}{2}} = \sum_{l=0}^{\infty} P_l(x)h^l .$$

So let $h = r$ and $x = \cos\theta$ to immediately get

$$G(\cos\theta, r) = (1-2r\cos\theta + r^2)^{-\frac{1}{2}} = \sum_{l=0}^{\infty} P_l(\cos\theta)r^l$$

$$V_{near}(r,\theta) = \frac{Q}{4\pi\varepsilon_0}\frac{1}{\sqrt{1-2r\cos\theta+r^2}} = \frac{Q}{4\pi\varepsilon_0}\sum_{l=0}^{\infty} P_l(\cos\theta)r^l ,$$

where near emphasizes that we are near the origin. Note that this solution separates into a sum of product of functions:

$$V_{near}(r,\theta) = \frac{Q}{4\pi\varepsilon_0}\sum_{l=0}^{\infty} P_l(\cos\theta)r^l = \sum_{l=0}^{\infty} f_l(r)g_l(\theta) .$$

We will take $A = \frac{Q}{4\pi\varepsilon_0}$ and write $f(r) = Ar^l$ and $g_l(\theta) = \sum_{l=0}^{\infty} P_l(\cos\theta)$

Now we switch gears and work from the other end, i.e., Laplace's equation.

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial V}{\partial r}) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial V}{\partial\theta}) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2} = 0$$

Note that we are in spherical coordinates here.

There is no ϕ dependence so we toss that part out.

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial V}{\partial r}) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial V}{\partial\theta}) = 0$$

Our potential is a function or r and θ, i.e., $V(r,\theta)$. We use the separation of variables trick now.

$$V(r,\theta) = f(r)g(\theta)$$

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \frac{\partial\left[f(r)g(\theta)\right]}{\partial r}) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}(\sin\theta \frac{\partial\left[f(r)g(\theta)\right]}{\partial\theta}) = 0$$

$$\nabla^2 V = \frac{1}{r^2}\frac{d}{dr}(r^2 \frac{df(r)}{dr})g(\theta) + \frac{1}{r^2 \sin\theta}\frac{d}{d\theta}(\sin\theta \frac{dg(\theta)}{d\theta})f(r) = 0$$

Notation is a little easier to visualize without the arguments.

$$\nabla^2 V = \frac{1}{r^2}\frac{d}{dr}(r^2 \frac{df}{dr})g + \frac{1}{r^2 \sin\theta}\frac{d}{d\theta}(\sin\theta \frac{dg}{d\theta})f = 0$$

We want to separate the "r-stuff" from the "θ-stuff." So we divide by fg. We have seen this trick before within the context of cylindrical coordinates.

$$\frac{1}{f}\frac{1}{r^2}\frac{d}{dr}(r^2 \frac{df}{dr}) + \frac{1}{g}\frac{1}{r^2 \sin\theta}\frac{d}{d\theta}(\sin\theta \frac{dg}{d\theta}) = 0$$

Proceed now to clear the r^2 from the second term.

$$\frac{1}{f}\frac{d}{dr}(r^2 \frac{df}{dr}) + \frac{1}{g}\frac{1}{\sin\theta}\frac{d}{d\theta}(\sin\theta \frac{dg}{d\theta}) = 0$$

Each must be a constant.

$$\frac{1}{f}\frac{d}{dr}(r^2 \frac{df}{dr}) = \lambda$$

$$\frac{1}{g}\frac{1}{\sin\theta}\frac{d}{d\theta}(\sin\theta\frac{dg}{d\theta}) = -\lambda$$

Let's solve the radial equation.

$$\frac{1}{f}\frac{d}{dr}(r^2\frac{df}{dr}) = \lambda$$

But we know from before that the radial solution near the origin is $f(r) = Ar^l$.

Using $f = Ar^l$ in $\frac{1}{f}\frac{d}{dr}(r^2\frac{df}{dr}) = \lambda$, gives us

$$\frac{1}{f}\frac{d}{dr}\left[r^2 Alr^{l-1}\right] = \lambda, \quad \frac{1}{f}\frac{d}{dr}\left[Alr^{l+1}\right] = \lambda, \quad \text{and}$$

$$\frac{1}{f}Al(l+1)r^l = \lambda.$$

Watch below how we verify that we indeed get a constant!

Dividing by f leads to $\frac{1}{Ar^l}Al(l+1)r^l = \lambda$ and $\lambda = l(l+1)$.

Our differential equation for g, $\frac{1}{g}\frac{1}{\sin\theta}\frac{d}{d\theta}(\sin\theta\frac{dg}{d\theta}) = -\lambda$, becomes

$$\frac{1}{g}\frac{1}{\sin\theta}\frac{d}{d\theta}(\sin\theta\frac{dg}{d\theta}) = -l(l+1).$$

$$\frac{d}{d\theta}(\sin\theta\frac{dg}{d\theta}) = -l(l+1)g\sin\theta$$

$$\cos\theta\frac{dg}{d\theta}+\sin\theta\frac{d^2g}{d\theta^2}=-l(l+1)g\sin\theta$$

$$\cos\theta\frac{dg}{d\theta}+\sin\theta\frac{d^2g}{d\theta^2}+l(l+1)g\sin\theta=0$$

$$\sin\theta\frac{d^2g}{d\theta^2}+\cos\theta\frac{dg}{d\theta}+l(l+1)g\sin\theta=0$$

Now take $x=\cos\theta$. Then

$$\frac{d}{d\theta}=\frac{dx}{d\theta}\frac{d}{dx}=-\sin\theta\frac{d}{dx}=-\sqrt{1-x^2}\,\frac{d}{dx}$$ and

$$\frac{d^2}{d\theta^2}=-\frac{dx}{d\theta}\frac{d}{dx}\left[\sqrt{1-x^2}\,\frac{d}{dx}\right]=-\frac{dx}{d\theta}\left[\frac{-x}{\sqrt{1-x^2}}\frac{d}{dx}+\sqrt{1-x^2}\,\frac{d^2}{dx^2}\right].$$

Substitute $\frac{dx}{d\theta}=-\sin\theta=-\sqrt{1-x^2}$ to get $\frac{d^2}{d\theta^2}=-x\frac{d}{dx}+(1-x^2)\frac{d^2}{dx^2}$.

Our differential equation for $g(\theta)=y(x)$ becomes

$$\sin\theta\left[-xy'+(1-x^2)y''\right]+\cos\theta(-\sqrt{1-x^2})y'+l(l+1)y\sin\theta=0$$

Since $\sin\theta=\sqrt{1-x^2}$, $-xy'+(1-x^2)y''+\cos\theta(-y')+l(l+1)y=0$.

Since $x=\cos\theta$, we arrive at $-xy'+(1-x^2)y''+x(-y')+l(l+1)y=0$

$$(1-x^2)y''-2xy'+l(l+1)y=0$$

This is Legendre's differential equation.

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The Legendre differential equation is

$$(1-x^2)y''-2xy'+l(l+1)y=0,$$

where $l=0,1,2,3,...$

The solutions are the Legendre polynomials $P_l(x)$.

We have our solutions without solving the Legendre differential equation with the "Method of Frobenius" as you usually do in quantum mechanics. Instead, we have our solutions from a power series expansion:

$$G(x,h)=(1-2hx+h^2)^{-\frac{1}{2}}=\sum_{l=0}^{\infty}P_l(x)h^l.$$

PM3 (Practice Problem). From your result $V_{dipole}(\vec{r})=\frac{1}{4\pi\varepsilon_0}\frac{\vec{p}\cdot\hat{r}}{r^2}$ of the previous problem, show that $\vec{E}=-\nabla V(\vec{r})$ in spherical coordinates leads to

$$\vec{E}(r,\theta)=\frac{p}{4\pi\varepsilon_0}\frac{1}{r^3}(2\cos\theta\,\hat{r}+\sin\theta\,\hat{\theta}).$$

Then show that your expression for the electric field vector can be written as

$$\vec{E}(r,\theta)=\frac{1}{4\pi\varepsilon_0}\frac{1}{r^3}\left[3(\vec{p}\cdot\hat{r})\hat{r}-\vec{p}\right].$$