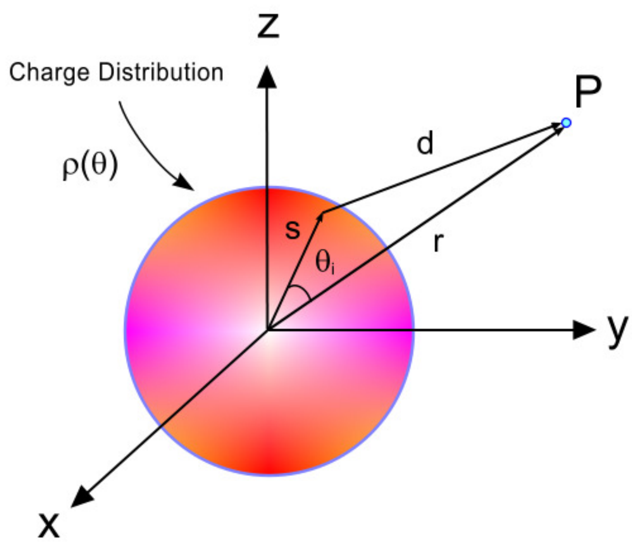
Electromagnetic Theory
Prof. Ruiz, UNC Asheville, doctorphys on YouTube
Chapter P Notes. Spherical Potentials



P1. Spherical Shells

We would like to consider potentials in free space produced by charge distributions which are spherical (round) in nature but have the general form

$$\rho = \rho(\theta)\,.$$

Working with this class of problems means solving Laplace's equation in spherical coordinates.

$$\nabla^2 V = 0$$

STEP 1. Reduced Laplacian. Since there is no dependence on ϕ, our potential will have the form $V = V(r,\theta)$. The Laplacian in spherical coordinates

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial V}{\partial r}) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\frac{\partial V}{\partial \theta}) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial \phi^2} = 0$$

simplifies to $\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial V}{\partial r}) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\frac{\partial V}{\partial \theta}) = 0$.

STEP 2. Separation of Variables. We know this works as demonstrated in the previous chapter.

$$V = V(r,\theta) = f(r)g(\theta)$$

STEP 3. We Know the General Angular Solution. From our Bonus Section in the previous class, we showed that the θ angular part separates out as Legendre's differential equation with the variable $\cos\theta$. The solutions are the Legendre polynomials

$$g_l(\theta) = P_l(\cos\theta)\,.$$

STEP 4. The Radial Equation. From the previous chapter we know that the radial equation separates out equal to the constant $\lambda = l(l+1)$, where $l = 0, 1, 2, 3, \ldots$.

$$\frac{1}{f}\frac{d}{dr}(r^2\frac{df}{dr}) = l(l+1)$$

At this point we do NOT want to make any assumptions on how large or small the radial variable might be. So we proceed in general. We want the general solution for f coming from

$$\frac{d}{dr}(r^2\frac{df}{dr}) = l(l+1)f$$

$$2rf' + r^2 f'' = l(l+1)f$$

$$r^2 f'' + 2rf' - l(l+1)f = 0$$

We again use the classic "Method of Frobenius" is to look for a solution in terms of a power series.

$$f(r) = \sum_{k=0}^{\infty} c_k r^k \qquad f'(r) = \sum_{k=1}^{\infty} k c_k r^{k-1} \qquad f''(r) = \sum_{k=2}^{\infty} k(k-1) c_k r^{k-2}$$

Note that one derivative kills the first term (the constant) and two derivatives kill off the first two terms. So we start k at 1 and at 2 in the above equations. But notice that we can start them with k = 0 anyway since the factors give zero for us anyway.

The result for is the following.

$$r^2 \sum_{k=0}^{\infty} k(k-1) c_k r^{k-2} + 2r \sum_{k=0}^{\infty} k c_k r^{k-1} - l(l+1) \sum_{k=0}^{\infty} c_k r^k = 0$$

We are again very lucky and arrive at a very simple result below - almost as simple as for the cylindrical Laplacian. As we noted in that other chapter, this does not happen in general as the powers of r are usually different in the various sums.

$$\sum_{k=0}^{\infty} k(k-1) c_k r^k + 2\sum_{k=0}^{\infty} k c_k r^k - m^2 \sum_{k=0}^{\infty} c_k r^k = 0$$

$$\sum_{k=0}^{\infty}\left[k(k-1)+2k-m^2\right]c_k r^k = 0$$

Since r is arbitrary and can be chosen at will, the part inside the brackets must vanish to make the equation true at all times.

$$k(k-1)+2k-l(l+1)=0$$

$$k^2-k+2k-l(l+1)=0$$

$$k^2+k-l(l+1)=0$$

It's time for fancy factoring. The answer is $(k-l)\left[k+(l+1)\right]=0$, with solutions

$$k_1 = l \quad \text{and} \quad k_2 = -(l+1)$$

PP1 (Practice Problem). Solve for these solutions the alternate way by solving the quadratic equation $k^2+k-l(l+1)=0$ since it is in the form $ax^2+bx+c=0$. The quadratic equation is used so often in physics that you should have it memorized,

$$k = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

Our solutions r^k therefore have the form r^l and $\frac{1}{r^{l+1}}$. The general solutions for f and g are below where l equals 0, 1, 2, 3, ...

$$f(r) = Ar^l + \frac{B}{r^{l+1}} \qquad g(\theta) = P_l(\cos\theta) \qquad V(r,\theta) = f(r)g(\theta)$$

We can write the general solution as a sum over all the l values.

$$V(r,\theta) = \sum_{l=0}^{\infty}\left[Ar^l + \frac{B_l}{r^{l+1}}\right]P_l(\cos\theta)$$

P2. A Spherical Shell Example

We consider a thin spherical shell with radius a and a boundary surface charge density

$$\sigma(\theta) = \sigma_0 \cos\theta \, .$$

The charge density is positive in the northern hemisphere $(0 \le \theta \le \pi / 2)$ and negative in the southern hemisphere $(\pi / 2 \le \theta \le \pi)$ with highest charge densities near the poles.

Note that this is an areal density. The total charge on the shell's surface is found by integrating over the entire surface. We can extract the surface area differential from the spherical volume element as seen in the very nice figure by Prof. Carl R. (Rod) Nave of the Department of Physics and Astronomy at Georgia State University. The figure comes from his popular HyperPhysics website. The volume element is

$$d\tau = r^2 \sin\theta dr d\theta d\phi \, .$$

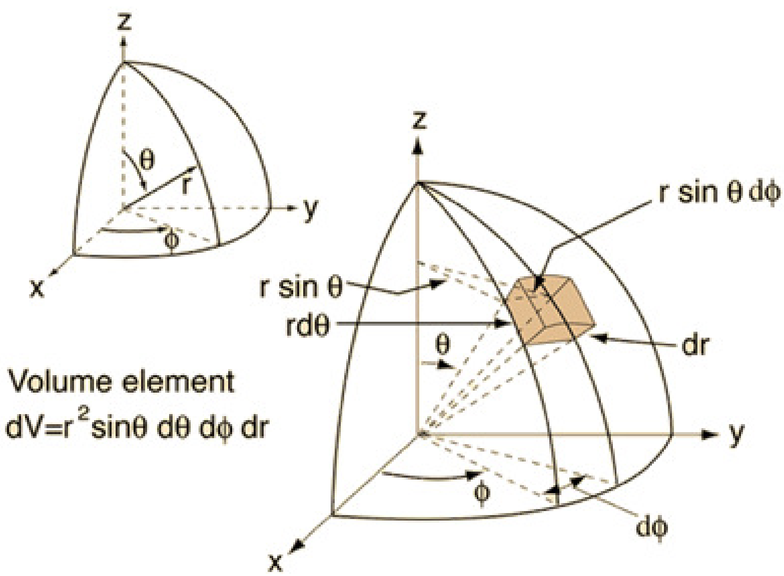


Figure Courtesy R. Nave
HyperPhysics

The patch of surface area is given by

$$(r d\theta)(r \sin\theta d\phi)$$

Since we are at the radius

$$r = a \, ,$$

we can write

$$dA = a^2 \sin\theta d\theta d\phi) \, .$$

The total charge is then $Q = \int_A \rho(\theta) dA$, which should integrate out to zero due to the positive and negative hemispheres. Let's set up the integral for practice.

$$Q = \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} (\sigma_0 \cos\theta) a^2 \sin\theta d\theta d\phi$$

PP2 (Practice Problem). Complete the integration for Q to show that the total charge is zero as expected for our specific charge density, i.e., show

$$Q = \sigma_0 a^2 \int_{\theta=0}^{\pi} \cos\theta \sin\theta d\theta \int_{\phi=0}^{2\pi} d\phi = 0 .$$

NOTE: You always want to be able to set up such integrals from scratch and work them out without integral tables when the integrands are fairly simple. You have already done similar integrals for spherical charge distributions in conjunction with Gauss's Law and cylindrical current densities related to Ampère's Law.

PP3 (Practice Problem). Use a symmetry argument considering even and odd functions over the integration range to establish that the integral in PP2 is zero by inspection. This is a clever shortcut.

We now proceed with the solution to Laplace's equation. Since the charge density does not depend on ϕ, the general solution is

$$V(r,\theta) = \sum_{l=0}^{\infty} \left[A_l r^l + \frac{B_l}{r^{l+1}} \right] P_l(\cos\theta) .$$

Now comes some nice boundary-condition physics. We cannot have "blow-ups" of the potential, i.e., the potential cannot run off to infinity. Therefore,

$$V_{in}(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) \text{, i.e., for } r < a ,$$

$$V_{out}(r,\theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta) \text{, i.e., for } r > a .$$

Remember that we can use these solutions to Laplace's equation because Laplace's equation is true inside and outside the shell region, i.e., in empty space.

Matching these at the boundary means $V_{in}(a,\theta) = V_{out}(a,\theta)$.

$$\sum_{l=0}^{\infty} A_l a^l P_l(\cos\theta) = \sum_{l=0}^{\infty} \frac{B_l}{a^{l+1}} P_l(\cos\theta)$$

$$\sum_{l=0}^{\infty}\left[A_l a^l - \frac{B_l}{a^{l+1}}\right]P_l(\cos\theta) = 0$$

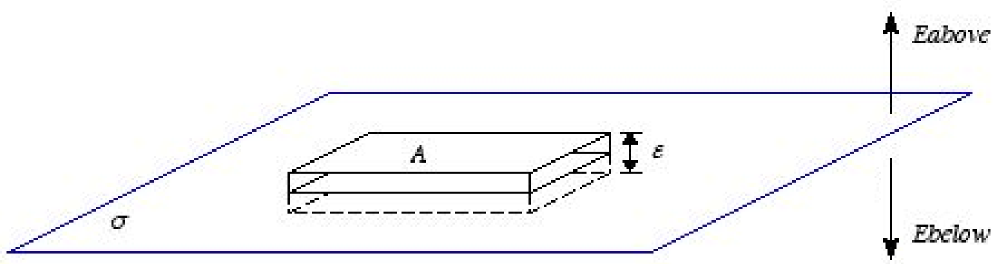
Using the arbitrary argument, what's inside the brackets must vanish.

Then, $A_l a^l = \dfrac{B_l}{a^{l+1}}$ and $B_l = A_l a^{2l+1}$.

$$V_{in}(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$$, i.e., for $r < a$,

$$V_{out}(r,\theta) = \sum_{l=0}^{\infty} \frac{A_l a^{2l+1}}{r^{l+1}} P_l(\cos\theta)$$, i.e., for $r > a$.

At the shell boundary we have to careful with the electric field since the electric field involves a derivative of the potential and we have a discontinuity at the shell boundary. Recall the calculation of the electric field for an infinite plane of charge.



Courtesy Prof. Frank L. H. Wolfs, Department of Physics
and Astronomy, University of Rochester, NY

Above the plane $\vec{E}_{above} = E\,\hat{k}$ and below $\vec{E}_{below} = -E\,\hat{k}$. Gauss's law $\oiint \vec{E}\cdot\vec{dA} = \dfrac{Q}{\varepsilon_0}$ then gives us $EA + EA = \dfrac{\sigma A}{\varepsilon_0}$ or $E + E = \dfrac{\sigma}{\varepsilon_0}$. We write this carefully as

$$E_{above} - E_{below} = \frac{\sigma}{\varepsilon_0}$$ since $E_{above} = E$ and $E_{below} = -E$.

Then for the derivative of the potential, we have

$$\left.\frac{dV}{dr}\right|_{above} - \left.\frac{dV}{dr}\right|_{below} = -\frac{\sigma}{\varepsilon_0}$$ since $E = -\frac{dV}{dr}$.

Applying this to our problem where "above" = "out" and "below" = "in,"

$$\left[\frac{dV_{out}}{dr} - \frac{dV_{in}}{dr}\right]_{r=a} = -\frac{\sigma(\theta)}{\varepsilon_0}.$$

With $V_{out}(r,\theta) = \sum_{l=0}^{\infty} \frac{A_l a^{2l+1}}{r^{l+1}} P_l(\cos\theta)$ and $V_{in}(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$,

\the equation $\left[\frac{dV_{out}}{dr} - \frac{dV_{in}}{dr}\right]_{r=a} = -\frac{\sigma(\theta)}{\varepsilon_0}$ becomes

$$\left[-\sum_{l=0}^{\infty}(l+1)\frac{A_l a^{2l+1}}{r^{l+2}} P_l(\cos\theta) - \sum_{l=0}^{\infty} l A_l r^{l-1} P_l(\cos\theta)\right]_{r=a} = -\frac{\sigma(\theta)}{\varepsilon_0}$$

$$-\sum_{l=0}^{\infty}(l+1)\frac{A_l a^{2l+1}}{a^{l+2}} P_l(\cos\theta) - \sum_{l=0}^{\infty} l A_l a^{l-1} P_l(\cos\theta) = -\frac{\sigma(\theta)}{\varepsilon_0}$$

$$-\sum_{l=0}^{\infty}(l+1) A_l a^{l-1} P_l(\cos\theta) - \sum_{l=0}^{\infty} l A_l a^{l-1} P_l(\cos\theta) = -\frac{\sigma(\theta)}{\varepsilon_0}$$

$$\sum_{l=0}^{\infty}(l+1) A_l a^{l-1} P_l(\cos\theta) + \sum_{l=0}^{\infty} l A_l a^{l-1} P_l(\cos\theta) = \frac{\sigma(\theta)}{\varepsilon_0}$$

$$\sum_{l=0}^{\infty}(2l+1) A_l a^{l-1} P_l(\cos\theta) = \frac{\sigma(\theta)}{\varepsilon_0}$$

Now, remember that $\sigma(\theta) = \sigma_0 \cos\theta$.

$$\sum_{l=0}^{\infty}(2l+1)A_l a^{l-1}P_l(\cos\theta)=\frac{\sigma_0\cos\theta}{\varepsilon_0}$$

Since $P_l(\cos\theta)=\cos\theta$, the $l=1$ term is the only nonzero term.

$$\left[(2l+1)A_l a^{l-1}P_l(\cos\theta)\right]_{l=1}=\frac{\sigma_0\cos\theta}{\varepsilon_0}$$

$$3A_1P_1(\cos\theta)=\frac{\sigma_0\cos\theta}{\varepsilon_0} \quad \text{and} \quad A_1=\frac{\sigma_0}{3\varepsilon_0}.$$

Summary:

$$V_{in}(r,\theta)=\sum_{l=0}^{\infty}A_l r^l P_l(\cos\theta) \text{ with the only nonzero } A_1=\frac{\sigma_0}{3\varepsilon_0}$$

$$V_{out}(r,\theta)=\sum_{l=0}^{\infty}\frac{A_l a^{2l+1}}{r^{l+1}}P_l(\cos\theta) \text{ with the only nonzero } A_1=\frac{\sigma_0}{3\varepsilon_0}$$

The solutions are

$$V_{in}(r,\theta)=A_1r^1P_1(\cos\theta)=\frac{\sigma_0}{3\varepsilon_0}r\cos\theta,$$

$$V_{out}(r,\theta)=\frac{A_1a^{2+1}}{r^{1+1}}P_1(\cos\theta)=\frac{\sigma_0}{3\varepsilon_0}a^3\frac{1}{r^2}\cos\theta.$$

The final forms are

$$V_{in}(r,\theta)=\frac{\sigma_0}{3\varepsilon_0}r\cos\theta \quad \text{and} \quad V_{out}(r,\theta)=\frac{\sigma_0}{3\varepsilon_0}\frac{a^3}{r^2}\cos\theta.$$

Note that the solutions indeed match at the boundary as we set them to earlier.

PP4 (Practice Problem). Take the negative gradient in spherical coordinates to find the electric fields inside and outside. Start with

$$\vec{E}(r,\theta)=-\nabla V(r,\theta)=\frac{\partial V}{\partial r}\hat{r}+\frac{1}{r}\frac{\partial V}{\partial\theta}\hat{\theta}+\frac{1}{r\sin\theta}\frac{\partial V}{\partial\phi}\hat{\phi}.$$