Electromagnetic Theory
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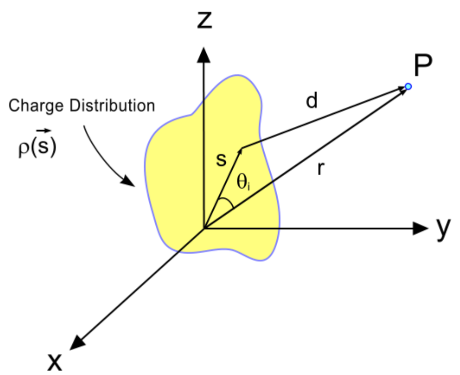
Chapter Q Notes. Method of Images

Q1. Insight into Multipoles



The first three terms in our multipole expansion from an earlier chapter are below. Note the Legendre parts.

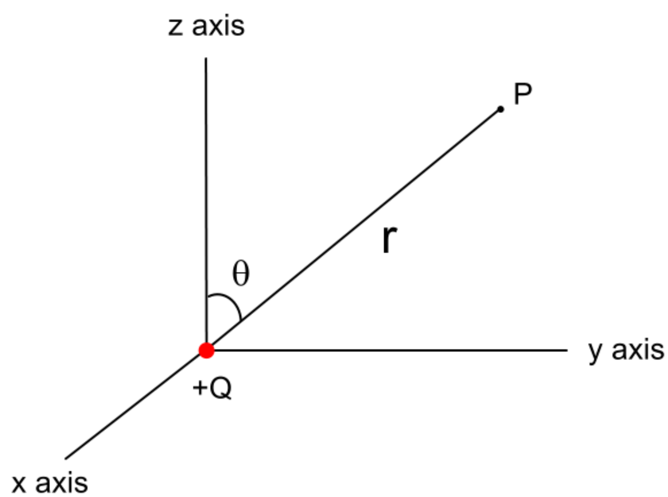
$$V_{monopole}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{1}{r}\int_\tau \rho(\vec{s})\, d^3\vec{s} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r}$$

$$V_{dipole}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^2}\int_\tau \cos\theta_i s\rho(\vec{s})\, d^3\vec{s}$$

$$V_{quadrupole}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^3}\int_\tau \frac{1}{2}(3\cos^2\theta_i - 1)s^2\rho(\vec{s})\, d^3\vec{s}$$

These come from $V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\sum_{l=0}^{\infty}\frac{1}{r^{l+1}}\int_\tau P_l(\cos\theta_i)s^l\rho(\vec{s})\, d^3\vec{s}$.

1. The Simplest Monopole - the Point Charge.



$$V_{mono}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r}$$

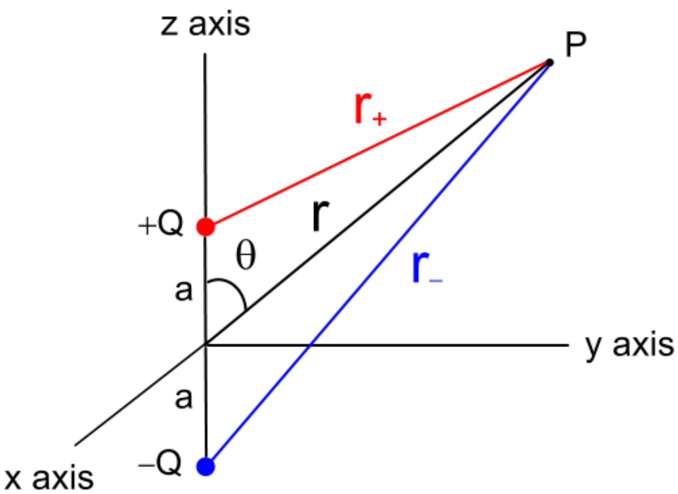
Note the 1/r dependence.

We can also write

$$V_{mono}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r}P_0(\cos\theta)$$

since the zeroth Legendre polynomial is 1.

2. The Simplest Dipole - Two Opposite Point Charges.



$$V_{dip}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\left[\frac{Q}{r_+} - \frac{Q}{r_-}\right]$$

$$r_+ = \sqrt{r^2 - 2ar\cos\theta + a^2}$$

$$r_- = \sqrt{r^2 + 2ar\cos\theta + a^2}$$

Use $(1+\varepsilon)^{-\frac{1}{2}} \approx 1 - \frac{1}{2}\varepsilon$.

$$r_+ = r\sqrt{1 - 2\frac{a}{r}\cos\theta + \frac{a^2}{r^2}} \qquad r_+ = r\sqrt{1 + 2\frac{a}{r}\cos\theta + \frac{a^2}{r^2}}$$

$$\frac{1}{r_+} = \frac{1}{r}(1 - 2\frac{a}{r}\cos\theta + \frac{a^2}{r^2})^{-1/2} \approx \frac{1}{r}(1 + \frac{a}{r}\cos\theta) \quad \text{to power} \quad \frac{1}{r^2}$$

$$\frac{1}{r_-} = \frac{1}{r}(1 + 2\frac{a}{r}\cos\theta + \frac{a^2}{r^2})^{-1/2} \approx \frac{1}{r}(1 - \frac{a}{r}\cos\theta) \quad \text{to power} \quad \frac{1}{r^2}$$

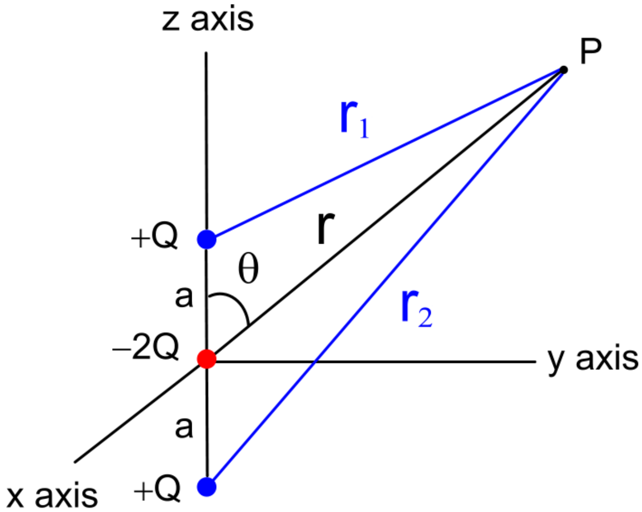
$$V_{dip}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r}\left[(1 + \frac{a}{r}\cos\theta) - (1 - \frac{a}{r}\cos\theta)\right] \quad \text{to power} \quad \frac{1}{r^2}$$

$$V_{dip}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r}\left[2\frac{a}{r}\cos\theta\right] \quad \text{to power} \quad \frac{1}{r^2} \quad \text{or} \quad r >> a$$

$$V_{dip}(\vec{r}) = \frac{Q}{2\pi\varepsilon_0}\frac{a}{r^2}\cos\theta = \frac{Q}{2\pi\varepsilon_0}\frac{a}{r^2}P_1(\cos\theta)$$

Note the $1/r^2$ dependence and the Legendre polynomial $P_1(\cos\theta) = \cos\theta$.
Note also that the monopole terms cancelled out and we obtained the next term!

3. The Linear Quadrupole - Four Point Charges (Two Opposite Dipoles).



$$V_{quad} = \frac{1}{4\pi\varepsilon_0}\left[-\frac{2Q}{r} + \frac{Q}{r_1} + \frac{Q}{r_2}\right]$$

$$r_1 = \sqrt{r^2 - 2ar\cos\theta + a^2}$$

$$r_2 = \sqrt{r^2 + 2ar\cos\theta + a^2}$$

In our expansion we must keep an extra term since we expect $1/r^3$ dependence,

Use $(1+\varepsilon)^{-\frac{1}{2}} \approx 1 - \frac{1}{2}\varepsilon - \frac{1}{2}(-\frac{3}{2})\frac{\varepsilon^2}{2!} = 1 - \frac{1}{2}\varepsilon + \frac{3}{8}\varepsilon^2$.

We want everything up to $1/r^3$ so part of ε^2 needs to be included.

$$\frac{1}{r_1} = \frac{1}{r}(1 - 2\frac{a}{r}\cos\theta + \frac{a^2}{r^2})^{-1/2} \approx \frac{1}{r}(1 + \frac{a}{r}\cos\theta - \frac{1}{2}\frac{a^2}{r^2} + \frac{3}{8}\frac{4a^2}{r^2}\cos^2\theta)$$

$$\frac{1}{r_2} = \frac{1}{r}(1 + 2\frac{a}{r}\cos\theta + \frac{a^2}{r^2})^{-1/2} \approx \frac{1}{r}(1 - \frac{a}{r}\cos\theta - \frac{1}{2}\frac{a^2}{r^2} + \frac{3}{8}\frac{4a^2}{r^2}\cos^2\theta)$$

$$\frac{4\pi\varepsilon_0}{Q}V_{quad} = -\frac{2}{r} + \left[\frac{1}{r_1} + \frac{1}{r_2}\right]$$

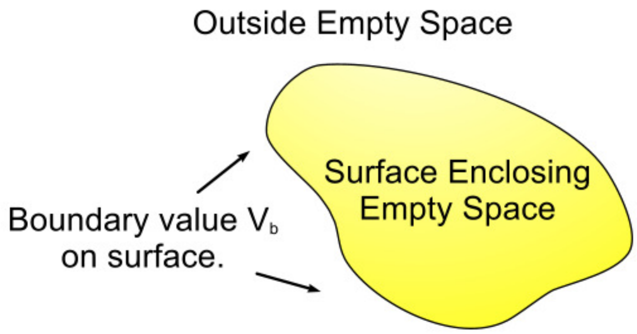
From above, the $\frac{1}{r}$ and $\frac{1}{r^2}$ terms cancel.

$$\frac{4\pi\varepsilon_0}{Q}V_{quad} = \frac{1}{r}\left[-\frac{a^2}{r^2} + \frac{3}{4}\frac{4a^2}{r^2}\cos^2\theta\right] = \frac{a^2}{r^3}(3\cos^2\theta - 1)$$

$$V_{quad} = \frac{Q}{4\pi\varepsilon_0}\frac{a^2}{r^3}(3\cos^2\theta - 1) = \frac{Q}{2\pi\varepsilon_0}\frac{a^2}{r^3}P_2(\cos\theta)$$

Note the $1/r^3$ dependence and appearance of the next Legendre polynomial.
Why did the monopole and dipoles terms cancel out and we obtain the next term?

Q2. Uniqueness Theorem



Laplace's equation

$$\nabla^2 V = 0$$

has a single unique solution.

Suppose there are two solutions g and h and each satisfies the boundary condition $g_b = h_b = b$ on the surface in the above figure.

Laplace's equation holds in the empty-space regions inside and outside the enclosed surface. Then $\nabla^2 g = \nabla^2 h = 0$ inside and outside the surface. Remember the following identity?

$$\nabla \cdot (f\vec{A}) = (\nabla f)\cdot \vec{A} + f\nabla \cdot \vec{A}$$

Here is a one-line derivation using Einstein's summation convention.

$$\nabla \cdot (f\vec{A}) = \frac{\partial}{\partial x_i}(fA_i) = \frac{\partial f}{\partial x_i}A_i + f\frac{\partial A_i}{\partial x_i} = (\nabla f)\cdot \vec{A} + f\nabla \cdot \vec{A}$$

Let's consider the more specific case where $\vec{A} = \nabla f$. Then we find the following.

$$\nabla \cdot (f\nabla f) = \nabla f \cdot \nabla f + f\nabla \cdot \nabla f$$

$$\boxed{\nabla \cdot (f\nabla f) = (\nabla f)^2 + f\nabla^2 f}$$

Now let $f = g - h$. The last term in the above equation is zero: $\nabla^2 f = \nabla^2 (g-h) = \nabla^2 g - \nabla^2 h = 0 - 0 = 0$ since g and h satisfy Laplace's equation. Our equation now simplies.

$$\boxed{\nabla \cdot (f\nabla f) = (\nabla f)^2}$$

Next we use the divergence theorem $\oiint \vec{F} \cdot \overrightarrow{da} = \iiint_{\tau} \nabla \cdot \vec{F} d\tau$ on the left side of the above equation. We want to go from volume to surface integral where the general vector field $\vec{F} = f \nabla f$ when applying the divergence theorem.

$$\iiint_{\tau} \nabla \cdot (f \nabla f) d\tau = \oiint (f \nabla f) \cdot \overrightarrow{da}$$

But since the right side is the surface integral, we are interested in values at the surface.

$$f_{surface} = (g-h)_{surface} = g_b - h_b = b - b = 0$$

These surface values come from our boundary conditions. Therefore,

$\iiint_{V} \nabla \cdot (f \nabla f) d\tau = 0$ and since $\nabla \cdot (f \nabla f) = (\nabla f)^2$, we are left with

$$\iiint_{V} (\nabla f)^2 d\tau = 0 .$$

At this point you might think we can use the arbitrary-volume rule and set what's inside the integral to zero. But we cannot use the arbitrary rule since the volume is specific. It is the volume enclosed by our surface. Now comes a very nice observation. Watch. We have a square as the integrand. That means only positive contributions since there are no imaginary numbers floating around for a potential, i.e., a real function.

There is no way to get zero for the integration unless $(\nabla f)^2 = 0$. This leads to $\nabla f = 0$, which means $f \equiv g - h = const$. But the constant has to be zero since we know that $f_{surface} = (g-h)_{surface} = 0$. If you know the value of a constant at any place, you are in. So $f \equiv g - h = 0$ always. The two solutions are identical: $g = h$. This is what we wanted to prove: that there is one unique solution. We call our unique solution for the potential by the standard letter V. There is only one unique potential:

$$V = g = h .$$

Q3. Method of Images

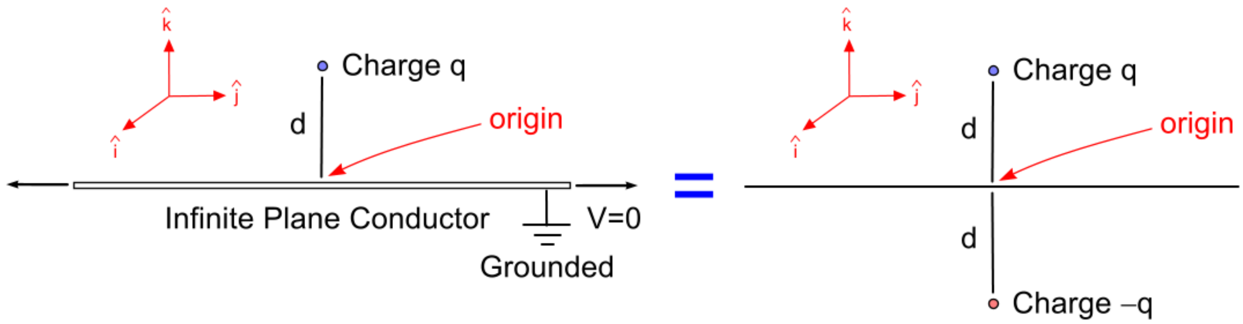
A conductor is a material, typically metal, in which charges can move. If we apply an electric field to a conductor, the charges in the conductor move to cancel out the electric field in the conductor.

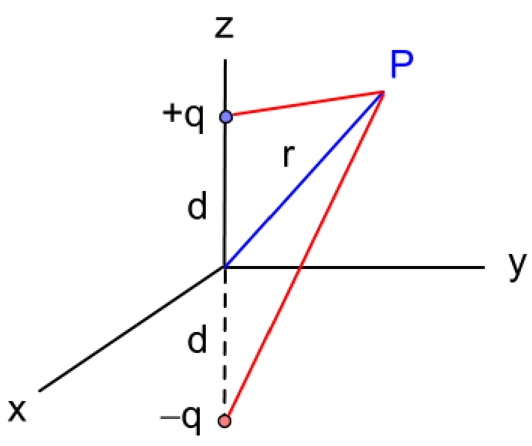
Hold a charge q a distance d above a grounded conducting plane. The charge is then at

$$(x, y, z) = (0, 0, d) .$$

For the grounded conductor, the potential everywhere is zero on the conductor. Therefore,

$$V(x, y, 0) = 0 .$$





The uniqueness theorem tells us if we can guess a potential that works, then that is a unique solution. So to get a potential zero in the x-y plane, we construct an alternate problem with a charge q above the plane and charge –q below.

The potential for the alternative problem has the same boundary conditions as our original problem with the single charge and grounded conducting plane.

The potential for the dipole arrangement is

$$V(x, y, z) = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{\sqrt{x^2 + y^2 + (z-d)^2}} - \frac{q}{\sqrt{x^2 + y^2 + (z+d)^2}}\right].$$

Boundary-Condition Check: $V(x, y, 0) = 0$ and $V_{far}(x, y, z) = 0$.

We can figure out something that at first glance appears very complicated: the charge density $\rho = \rho(x, y)$ on the conductor. Remember our rule for determining charge density on a sphere?

$$\left.\frac{dV}{dr}\right|_{above} - \left.\frac{dV}{dr}\right|_{below} = -\frac{\sigma}{\varepsilon_0}$$

We would like to be able to write this rule for any shaped surface, so we replace these radial derivatives with partial derivatives along the normal.

$$\left.\frac{\partial V}{\partial n}\right|_{above} - \left.\frac{\partial V}{\partial n}\right|_{below} = -\frac{\sigma}{\varepsilon_0}$$

For a conductor $\left.\frac{\partial V}{\partial n}\right|_{below} = 0$ since there is no electric field in a conductor. If an electric field is applied to a conductor, the charges inside the conductor move to cancel the electric field. For the conductor we write the following equation for the surface charge density.

$$\sigma = -\varepsilon_0 \left.\frac{\partial V}{\partial n}\right|_{above} = -\varepsilon_0 \lim_{z \to 0+} \frac{\partial V}{\partial z} = -\varepsilon_0 \left.\frac{\partial V}{\partial z}\right|_{z=0}$$

$$V(x, y, z) = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{\sqrt{x^2 + y^2 + (z-d)^2}} - \frac{q}{\sqrt{x^2 + y^2 + (z+d)^2}}\right]$$

$$4\pi\varepsilon_0 \frac{\partial V}{\partial z} = \frac{(-1/2)q\left[2(z-d)\right]}{\left[x^2 + y^2 + (z-d)^2\right]^{-3/2}} - \frac{(-1/2)q\left[2(z+d)\right]}{\left[x^2 + y^2 + (z+d)^2\right]^{-3/2}}$$

$$4\pi\varepsilon_0 \frac{\partial V}{\partial z} = \frac{-q(z-d)}{\left[x^2 + y^2 + (z-d)^2\right]^{-3/2}} + \frac{q(z+d)}{\left[x^2 + y^2 + (z+d)^2\right]^{-3/2}}$$

$$4\pi\varepsilon_0 \left.\frac{\partial V(x,y,z)}{\partial z}\right|_{z=0} = \frac{2qd}{(x^2+y^2+d^2)^{-3/2}}$$

$$\left.\frac{\partial V(x,y,z)}{\partial z}\right|_{z=0} = \frac{1}{4\pi\varepsilon_0}\frac{2qd}{(x^2+y^2+d^2)^{-3/2}}$$

$$\sigma = -\varepsilon_0 \left.\frac{\partial V}{\partial z}\right|_{z=0} = -\varepsilon_0 \frac{1}{4\pi\varepsilon_0}\frac{2qd}{(x^2+y^2+d^2)^{-3/2}}$$

$$\sigma(x,y) = -\frac{1}{2\pi}\frac{qd}{(x^2+y^2+d^2)^{-3/2}}$$

The induced surface charge is negative.

Let's integrate to find the total charge on the conductor's surface. It is best to use polar coordinates (ρ,ϕ), where $\rho^2 = x^2 + y^2$.

$$\sigma = -\frac{1}{2\pi}\frac{qd}{(x^2+y^2+d^2)^{-3/2}} = -\frac{1}{2\pi}\frac{qd}{(\rho^2+d^2)^{-3/2}}$$

$$Q = \int_{\rho=0}^{\infty}\int_{\phi=0}^{2\pi} \sigma\rho d\rho d\phi = -\frac{qd}{2\pi}\int_{\rho=0}^{\infty}\int_{\phi=0}^{2\pi}\frac{1}{(\rho^2+d^2)^{-3/2}}\rho d\rho d\phi$$

$$Q = -\frac{qd}{2\pi}\int_{\phi=0}^{2\pi} d\phi \int_{\rho=0}^{\infty}\frac{1}{(\rho^2+d^2)^{-3/2}}\rho d\rho$$

$$Q = -\frac{qd}{2\pi}2\pi\left[\int_{\rho=0}^{\infty}\frac{1}{(\rho^2+d^2)^{-3/2}}2\rho d\rho\right]\frac{1}{2}$$

$$Q = qd\left.\frac{1}{\sqrt{\rho^2+d^2}}\right|_0^{\infty} = qd(0-\frac{1}{d}) = -q$$