Electromagnetic Theory
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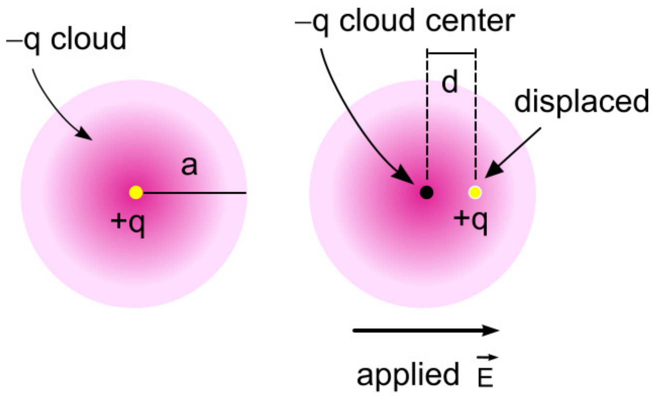
Chapter R Notes. Dielectrics

R1. Dipole Moments



1. Induced Dipole. We have already seen dipoles in our course. Here we look at an induced dipole where we use a simple model for an atom. The atom has +q at the center and an electron cloud of total −q.

Then we apply an external electric field. The positive center displaces slightly, which we exaggerate in the figure. The displacement is to the right since the electric field is applied to the right.

The negative electric field at the position of the displaced positive charge pulls it back to the left. The forces cancel and we have equilibrium. We find the strength of the negative electric field using Gauss's Law in the interior. We will take the charge density to be constant. **Do not worry about memorizing the new equations in this chapter.**

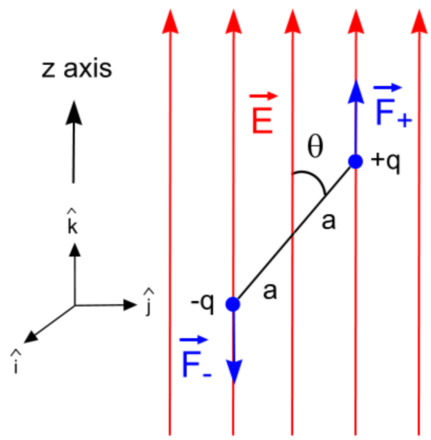
$$\oiint \vec{E}_{cloud} \cdot \vec{dA} = \frac{Q_{in}}{\varepsilon_0} \qquad E_{cloud} 4\pi d^2 = -\frac{q}{\varepsilon_0}\frac{d^3}{a^3} \text{ (constant density model)}$$

$$E_{cloud} = -\frac{q}{4\pi\varepsilon_0}\frac{d}{a^3} \quad \text{and} \quad E_{cloud} = E_{applied} = \frac{q}{4\pi\varepsilon_0}\frac{d}{a^3}$$

The total force on the positive-core charge is zero. Earlier we defined the dipole moment as the product of the charge and the distance between the two charges: $p = qd$. Here our distance is between the positive charge and the center of the negative electron cloud. The dipole direction is taken from the negative charge to the plus charge. Our dipole field lines up with the applied electric field:

$$\vec{E} = \frac{\vec{p}}{4\pi\varepsilon_0 a^3} \quad \text{and} \quad \boxed{\vec{p} = \alpha\vec{E}}, \text{ where } \alpha = 4\pi\varepsilon_0 a^3 \text{ for this super simple model.}$$

If the applied field is not too strong, a linear relation often works but with different α values depending on the situation.



2. The Torque on a Dipole. We apply an electric field to an already-existing dipole in this case. We apply the torque equation for each charge.

$$\vec{\tau} = \vec{r}_{+} \times \overrightarrow{F_{+}} + \vec{r}_{-} \times \overrightarrow{F_{-}}$$

$$\vec{\tau} = 2\vec{r} \times \vec{F}$$

$$\vec{\tau} = 2aF \sin\theta \hat{i} \quad \text{and} \quad \vec{\tau} = 2aqE \sin\theta \hat{i}$$

$$\vec{\tau} = pE \sin\theta \hat{i}$$

$$\boxed{\vec{\tau} = \vec{p} \times \vec{E}}$$

3. Nonuniform External Force on Dipole. We now compare what we have in the above figure to a case where the external electric field is not constant.

For uniform external $\vec{E}$ field: $\vec{F} = \overrightarrow{F_{+}} + \overrightarrow{F_{-}} = q\vec{E} - q\vec{E} = q(\vec{E} - \vec{E}) = 0$

For nonuniform $\vec{E}$ field: $\vec{F} = \overrightarrow{F_{+}} + \overrightarrow{F_{-}} = q\vec{E}(z + \Delta z) - q\vec{E}(z) = q\overrightarrow{\Delta E} \neq 0$

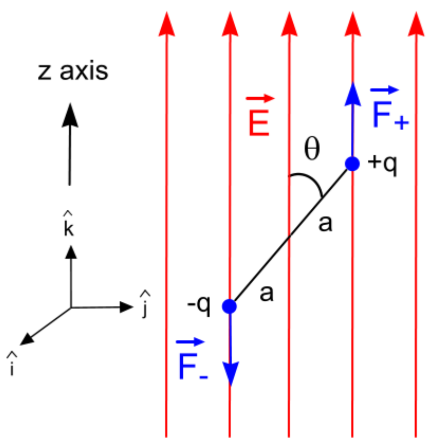
We can write this as $\Delta E = \dfrac{\partial E_z}{\partial z} \Delta z = \dfrac{\partial E_z}{\partial z}(2a\cos\theta)$.

$$F = q\Delta E = (2aq\cos\theta)\frac{\partial E_z}{\partial z} = \vec{p} \cdot \hat{k} \frac{\partial E_z}{\partial z}$$

$$F_z = \vec{p} \cdot \hat{k} \frac{\partial E_z}{\partial z}$$

$$\boxed{\vec{F} = (\vec{p} \cdot \nabla)\vec{E}}$$

It is safe to write the equation as we have with $\vec{p}$ at the left in case $\vec{p}$ isn't constant.



4. Dipole Potential Energy. We want to rotate the dipole clockwise as if winding a spring to let it go. The work done is then

$$U = W = \int \tau d\theta \,.$$

$$U = 2aF \int \sin\theta d\theta = 2aqE \int \sin\theta d\theta$$

$$U = -2aqE\cos\theta + const$$

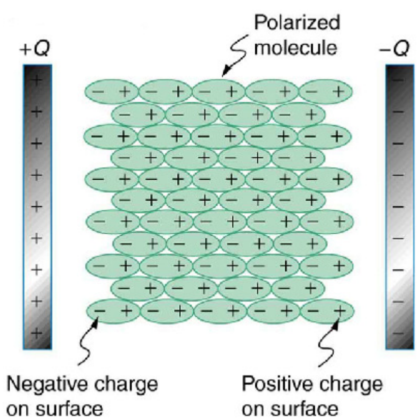
$$U = -2aqE\cos\theta + const$$ The constant is chosen to be zero.

$$U = -2aqE\cos\theta$$

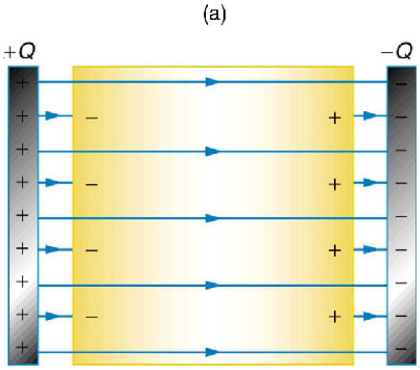
$$\boxed{U = -\vec{p}\cdot\vec{E}}$$

Which angle gives zero potential energy by our zero-point convention?

R2. The Polarization Vector



(a)



(b)

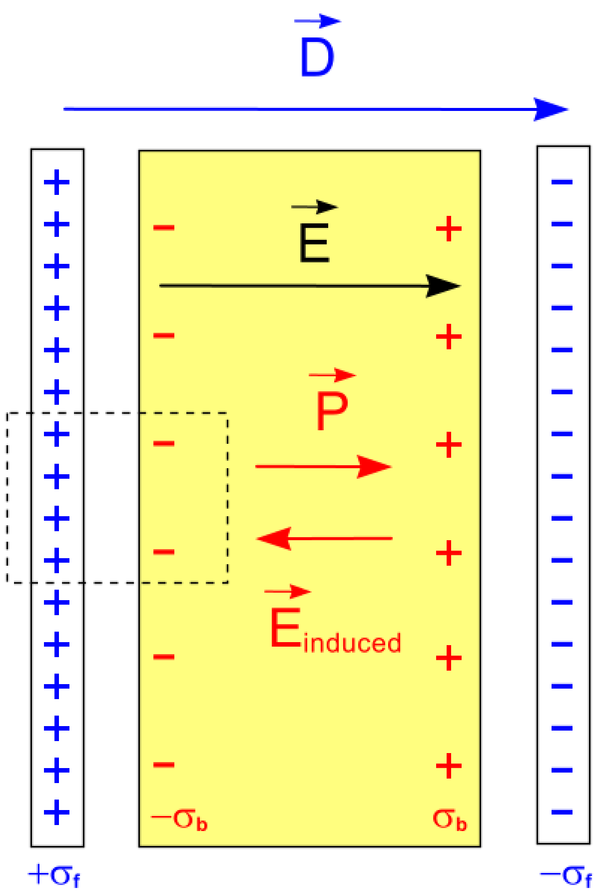
Figure Courtesy Rice University

There are two basic types of common materials as we continue to apply our physics to engineering-related problems: the conductor and non-conductor. The material either conducts electricity or it does not. The non-conductor is called an insulator or a dielectric.

See the (a) part of the left figure for an insulator placed between charged capacitor plates. An induced polarization occurs at the molecular level. The charges in the material align such that opposites are attracted to the plates on each side. The net bound charge appears at the extreme sides of the dielectric.

See the (b) part of the left figure for the electric field lines set up by the capacitor plates. These get weakened due to the opposite electric field induced in the dielectric slab.

From earlier in course, the electric field E between the plates (see figure below) is $\frac{\sigma}{\varepsilon_0}$, where σ is the total charge density. We can write the total charge density as a sum of the charge density on the plate plus the induced charge in the dielectric: $\sigma = \sigma_f - \sigma_b$, where the induced charge $\sigma_b > 0$ in our figure. Then



$$\varepsilon_0 E = \sigma_f - \sigma_b .$$

The "f" stands for free and "b" for bound. We define $\vec{D}$ and $\vec{P}$ so that

$$\varepsilon_0 \vec{E} = \vec{D} - \vec{P} .$$

Note the missing ε_0 in the D and P definitions. Also note that the arrow for P must be drawn from negative to positive.

The polarization is the combined effect of the little dipoles lining up in the material to provide for the induced charge. Our equation $\varepsilon_0 \vec{E} = \vec{D} - \vec{P}$ is often written as follow.

$$\boxed{\vec{D} = \varepsilon_0 \vec{E} + \vec{P}}$$

In electrostatics $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = 0$ since "statics" means no functions of time. Note that in general $\nabla \times \vec{D} = \varepsilon_0 \nabla \times \vec{E} + \nabla \times \vec{P} = \nabla \times \vec{P} \neq 0$ due to possible complicated dipole configurations.

R3. Dielectric Constant

1. Susceptibility. For a linear dielectric, the induced polarization is proportional to the net electric field. The proportionality constant is the electric susceptibility.

$$\boxed{\vec{P} \equiv \varepsilon_0 \chi_e \vec{E}}$$

2. Relative Permittivity. From $\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$ we find

$$\vec{D} = \varepsilon_0 \vec{E} + \varepsilon_0 \chi_e \vec{E} = \varepsilon_0 (1 + \chi_e) \vec{E} .$$

The permittivity ε of the dielectric can be defined to simplify the above constants.

$$\boxed{\vec{D} = \varepsilon \vec{E}}$$

$$\boxed{\varepsilon = \varepsilon_0 (1 + \chi_e)}$$ With no polarization: $\vec{D} = \varepsilon_0 \vec{E}$.

The relative permittivity ε_r is defined next. We will see that It is also the dielectric constant.

$$\boxed{\varepsilon_r \equiv \frac{\varepsilon}{\varepsilon_0} = 1 + \chi_e}$$

3. Dielectric Constant. We will shortly review how to arrive these capacitance formulas for the parallel-plate capacitor, cylindrical capacitor, and spherical capacitor.

$$C_{parallel} = \frac{\varepsilon_0 A}{d} \qquad C_{cylindrical} = \frac{2\pi\varepsilon_0 l}{\ln(a/b)} \qquad C_{spherical} = 4\pi\varepsilon_0 \frac{ab}{b-a}$$

For now just note that ε_0 appears in the numerator for each. These results are for vacuum. If we insert a dielectric into the plates then we replace ε_0 with ε . In general, we can write. To replace: divide out the ε_0 and then multiply by ε .

$$C = \frac{\varepsilon}{\varepsilon_0} C_0 \qquad C = \varepsilon_r C_0$$

The dielectric constant κ is defined to be $\kappa \equiv \varepsilon_r$. The capacitance increases by this value $\kappa = \varepsilon_r = (1+\chi_e)$

$$\boxed{C = (1+\chi_e)C_0}$$

The proportionality factor, the dielectric constant, is our relative permittivity.

$$\boxed{\kappa = \varepsilon_r = \frac{\varepsilon}{\varepsilon_0}} \quad \boxed{C = \kappa C_0} \quad \boxed{C = (1+\chi_e)C_0} \quad \boxed{C = \varepsilon_r C_0}$$

Too many formulas!

At this point it's good to review our capacitance formulas where the capacitor has area A and plate separation d. First, let's review the definition of capacitance:

$$C \equiv \frac{Q}{V},$$

where +Q charge is on one plate and -Q on the other with the applied voltage V. You can see why this is the definition for a capacity by considering a constant V: if more Q can be put on the plates, then you have greater capacity to hold charge.

Between two large plates the electric field is a constant:

$$E = \frac{\sigma}{\varepsilon_0} \quad \text{and} \quad V = Ed .$$

These relations give the famous capacitance equation for parallel plates:

$$C \equiv \frac{Q}{V} = \frac{Q}{Ed} = \frac{\sigma A}{(\sigma / \varepsilon_0)d} = \frac{\varepsilon_0 A}{d} .$$

Inserting the dielectric gives us $C = \kappa C_0$. So we can write the more general result:

$$C = \frac{\kappa \varepsilon_0 A}{d} \quad \text{or} \quad C = \frac{\varepsilon A}{d} .$$

The last equation is a convenient general expression that depends on the permittivity of the medium in the capacitor.

Energy formulas. The energy in building up charge on a capacitor is given by

$$U = \int_0^Q V dq = \frac{1}{C}\int_0^Q q dq = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}QV = \frac{1}{2}CV^2, \text{ using } C = \frac{Q}{V}.$$

Energy per unit volume:

$$\frac{U}{\tau} = \frac{1}{2}\frac{CV^2}{Ad} = \frac{1}{2}\frac{C(Ed)^2}{Ad} = \frac{CE^2 d}{2A} = \frac{1}{2}\frac{\varepsilon A}{d}E^2\frac{d}{A} = \frac{1}{2}\varepsilon E^2.$$

Summary: $\boxed{U = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}QV = \frac{1}{2}CV^2}$ $\boxed{\frac{U}{\tau} = \frac{1}{2}\varepsilon E^2}$

R4. Capacitances for Common Geometries

Here is a prescription to find capacitances for symmetric capacitor configurations.

1) Use **Gauss's Law** to find the electric field between the plates. $\oiint \vec{E}\cdot\vec{dA} = \frac{Q}{\varepsilon_0}$

Result for Two Parallel Plates: $E = \frac{\sigma}{\varepsilon_0}$

2) Integrate to find the **Potential** and take the absolute value. $V = -\int \vec{E}\cdot\vec{dr}$

Result for Two Parallel Plates a Distance d apart: $V = Ed$

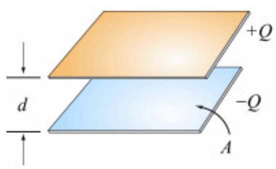
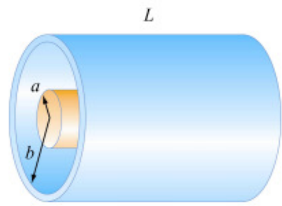
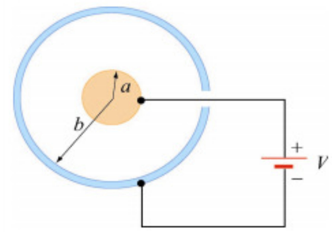
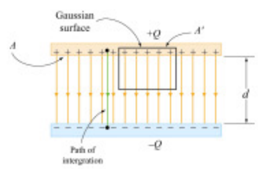
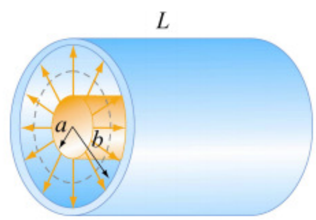
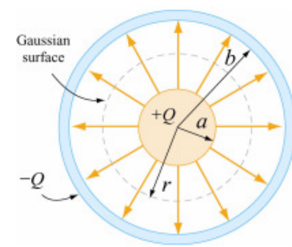
3) Use the definition of **Capacitance**. $C = \frac{Q}{V}$

Result for Two Parallel Plates: $C = \frac{Q}{V} = \frac{\sigma A}{Ed} = \frac{\sigma A}{(\sigma/\varepsilon_0)d} = \frac{\varepsilon_0 A}{d}$

PR1 (Practice Problem). Derive the formula for the cylindrical capacitor with inner radius a, outer radius b, and length l: $C = \dfrac{2\pi\varepsilon_0 l}{\ln(a/b)}$. See next page for solution.

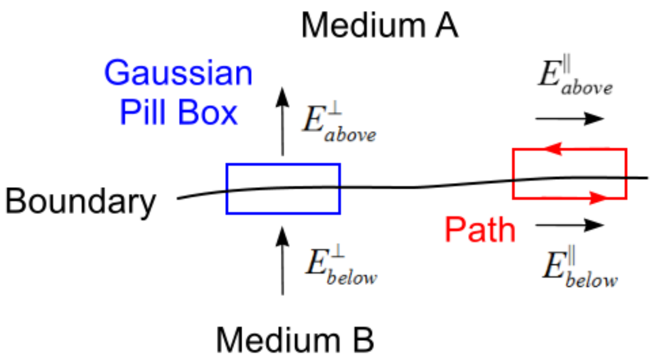
PR2 (Practice Problem). Derive the formula for the spherical; capacitor with inner radius a and outer radius b: $C = 4\pi\varepsilon_0 \dfrac{ab}{b-a}$. See next page for solution.

Courtesy MIT Open Courseware, Physics 8.02 Electricity and Magnetism

Capacitors	**Parallel-plate**	**Cylindrical**	**Spherical**
Figure			
(1) Identify the direction of the electric field using symmetry			
(2) Calculate electric field everywhere	$\oiint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = EA = \frac{Q}{\varepsilon_0}$ $E = \frac{Q}{A\varepsilon_0} = \frac{\sigma}{\varepsilon_0}$	$\oiint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = E(2\pi r l) = \frac{Q}{\varepsilon_0}$ $E = \frac{\lambda}{2\pi\varepsilon_0 r}$	$\oiint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = E_r(4\pi r^2) = \frac{Q}{\varepsilon_0}$ $E_r = \frac{1}{4\pi\varepsilon_o}\frac{Q}{r^2}$
(3) Compute the electric potential difference ΔV	$\Delta V = V_- - V_+ = -\int_+^- \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}}$ $= -Ed$	$\Delta V = V_b - V_a = -\int_a^b E_r dr$ $= -\frac{\lambda}{2\pi\varepsilon_0}\ln\left(\frac{b}{a}\right)$	$\Delta V = V_b - V_a = -\int_a^b E_r dr$ $= -\frac{Q}{4\pi\varepsilon_0}\left(\frac{b-a}{ab}\right)$
(4) Calculate C using $C = Q/\lvert\Delta V\rvert$	$C = \frac{\varepsilon_0 A}{d}$	$C = \frac{2\pi\varepsilon_0 l}{\ln(b/a)}$	$C = 4\pi\varepsilon_0\left(\frac{ab}{b-a}\right)$

R5. Boundary Conditions

We consider a boundary between two media.



The first Maxwell equation gives us a general boundary condition.

$$\oiint \vec{E} \cdot \vec{dA} = \frac{Q_{in}}{\varepsilon_0}$$

$$E^{\perp}_{above} A - E^{\perp}_{below} A = \frac{\sigma A}{\varepsilon_0}$$

$$E^{\perp}_{above} - E^{\perp}_{below} = \frac{\sigma}{\varepsilon_0}$$

The Gaussian pill box is our guide. Next we consider a loop path. Faraday's Law.

$$\oint \vec{E} \cdot \vec{dl} = -\frac{d\Phi_B}{dt}$$ reduces to $$\oint \vec{E} \cdot \vec{dl} = 0$$ since

in electrostatics, i.e., nothing changes. This gives a general electrostatic boundary condition for the parallel component of the electric field.

$$E^{\parallel}_{below} l - E^{\parallel}_{above} l = 0 \qquad E^{\parallel}_{above} = E^{\parallel}_{below}$$

Summary:

$$\boxed{E^{\perp}_{above} - E^{\perp}_{below} = \frac{\sigma}{\varepsilon_0}} \qquad \boxed{E^{\parallel}_{above} = E^{\parallel}_{below}}$$

The equations $\rho_f \equiv \nabla \cdot \vec{D}$ and $\nabla \cdot \vec{P} = -\rho_b$ lead to the following.

$$\boxed{D^{\perp}_{above} - D^{\perp}_{below} = \sigma_f} \qquad \boxed{P^{\perp}_{above} - P^{\perp}_{below} = \sigma_b}$$

Since we have no loop integral equations similar to $\oint \vec{E} \cdot \vec{dl} = 0$, we can work from the $\vec{E}$ loop-equation for the electric field and $\varepsilon_0 \vec{E} = \vec{D} - \vec{P}$ to get our next equation.

$$\boxed{D^{\parallel}_{above} - P^{\parallel}_{above} = D^{\parallel}_{below} - P^{\parallel}_{below}}$$