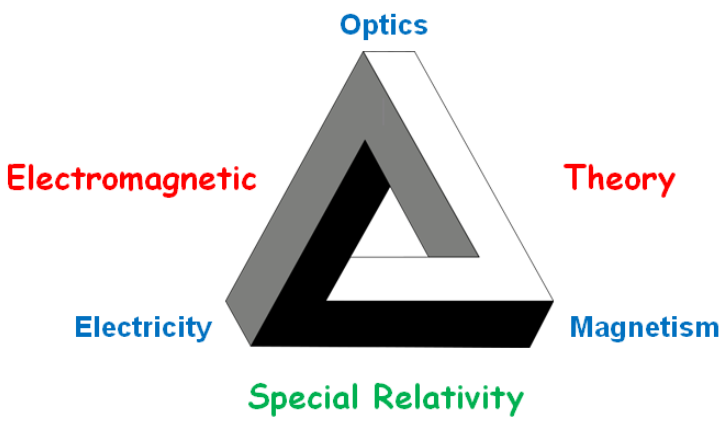
Electromagnetic Theory (EMT)
Prof. Ruiz, UNC Asheville, doctorphys on YouTube
Chapter S Notes. Lorentz Force Law

S1. EMT and Other Physics Courses

We have seen the figure below with its triad of courses: Optics, Electromagnetic Theory, and Special Relativity (Modern Physics).



We have also seen a connection to quantum mechanics through our deep analysis of Laplace's equation and Poisson's equation:

$$\nabla^2 V = 0 \qquad \text{and}$$

$$\nabla^2 V = -\frac{\rho}{\varepsilon_0} .$$

The Laplacian appears in the Schrödinger equation:

$$-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = E\psi .$$

In spherical coordinates, we found for our Laplacian solutions of the form

$$V(r,\theta) = \sum_{l=0}^{\infty}\left[Ar^l + \frac{B_l}{r^{l+1}}\right]P_l(\cos\theta) .$$

In quantum mechanics things are more complicated. For spherical problems, the associated Legendre polynomials $P_l^m(\theta)$ appear in the angular part: $Y_l^m(\theta,\phi) = N_{lm}P_l^m(\theta)e^{im\phi}$, where N_{lm} are normalization constants. The functions $Y_l^m(\theta,\phi)$ are called spherical harmonics. The general solution for spherical problems is a wave function of the following form.

$$\psi_{nlm}(r,\theta,\phi) = R_{nl}(r)Y_l^m(\theta,\phi)$$

Today we add a fifth course to the list: Classical Mechanics. Mechanics especially connects to our course via the Lorentz Force Law.

Electromagnetic Theory (The Maxwell Equations)
Optics (Electromagnetic Waves as a Solution to the Free-Space Maxwell Equations)
Special Relativity (Invariance of the Maxwell Equations)
Quantum Mechanics (The Mathematics of the Laplacian)
Classical Mechanics (The Lorentz Force Law)

Other connections with mechanics include the following:

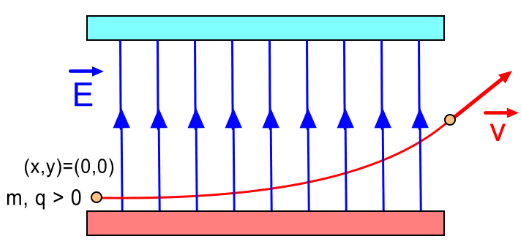
The Concept of the Potential
Kinetic and Potential Energy
Mechanical Energy converted to Electrical Energy

The Lorentz force law is perhaps the best since we can combine Newton's Law with it.

Mechanics	Electromagnetic Theory
$\vec{F} = m\vec{a}$	$\vec{F} = q(\vec{E} + \vec{v}\times\vec{B})$
Newton's 2nd Law	Lorentz Force Law

We have already seen the following two simple cases.

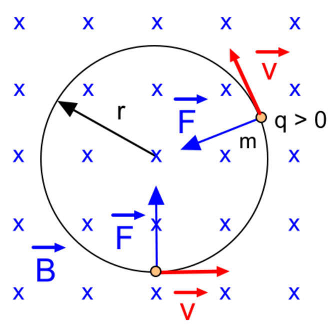
$$\vec{F} = q\vec{E}$$



$$x(t) = v_0 t$$

$$y(t) = \frac{qE}{2m}t^2$$

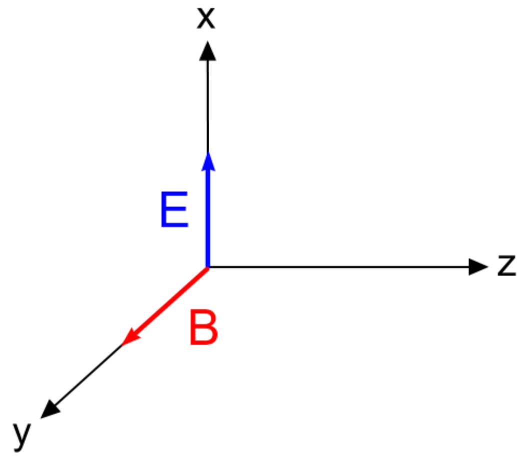
$$\vec{F} = q\vec{v}\times\vec{B}$$



$$\frac{mv^2}{r} = qvB \quad \text{and} \quad \omega = \frac{v}{r} = \frac{qB}{m}$$

In the next section we consider the simplest problem where there is both a constant electric field and a constant magnetic field.

S2. Lorentz Force



Place a charge q at the origin so that it is at rest at t = 0. What happens?

GIVEN: $\vec{E} = E\hat{i}$ and $\vec{B} = B\hat{j}$

Charge q with $\vec{r}(0) = 0$ and $\vec{v}(0) = 0$.

SOLUTION: We start the solution by combining Newton's Second Law with the Lorentz Force Law.

$$\vec{F} = m\vec{a} \qquad \vec{F} = q(\vec{E} + \vec{v}\times\vec{B})$$

We will use the notation where a dot means a derivative with respect to time. For the x-variable we have $v_x = \dot{x}$ and $a_x = \ddot{x}$.

$$\vec{F} = q(\vec{E} + \vec{v}\times\vec{B}) = qE\hat{i} + q\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dot{x} & \dot{y} & \dot{z} \\ 0 & B & 0 \end{vmatrix} = qE\hat{i} + q\hat{i}(-B\dot{z}) + q\hat{k}(\dot{x}B)$$

Combining with Newton's equations we arrive at three differential equations.

$$m\ddot{x} = qE - qB\dot{z}$$

$$m\ddot{y} = 0$$

$$m\ddot{z} = qB\dot{x}$$

The y-equation is the easiest. The solution to $\ddot{y} = 0$ gives $\dot{y}(t) = c_1$, where c_1 is the constant of integration. Then.

$$y(t) = c_0 + c_1 t\ .$$

But $c_0 = y(0) = 0$ and $c_1 = \dot{y}(0) = 0$.

Therefore, $y(t) = 0$. There is no motion along the y-axis.

This leaves $m\ddot{x} = qE - qB\dot{z}$ and $m\ddot{z} = q\dot{x}B$. Divide by the mass.

$$\ddot{x} = \frac{qE}{m} - \frac{qB}{m}\dot{z} \qquad \ddot{z} = \frac{qB}{m}\dot{x}$$

Let $\omega = \frac{qB}{m}$.

$$\ddot{x} = \omega\frac{E}{B} - \omega\dot{z} \qquad \ddot{z} = \omega\dot{x}$$

The trick now is to differential x with respect to time to obtain a term with $\ddot{z}$ which we then get rid of using our second equation, i.e., $\ddot{z} = \omega\dot{x}$.

$$\dddot{x} = -\omega\ddot{z}$$

$$\dddot{x} = -\omega^2\dot{x}$$

When you take two derivatives and get the same thing back with a minus sign out solutions are the cosine and sine.

$$\dot{x}(t) = a\cos\omega t + b\sin\omega t$$, where a and b are constants.

The initial condition for the speed is $\dot{x}(0) = a = 0$, which takes care of one constant.

Integrating

$$\dot{x}(t) = b\sin\omega t$$

leads to

$$x(t) = -\frac{b}{\omega}\cos\omega t + c$$, where c is a constant.

From the initial condition $x(0) = -\frac{b}{\omega} + c = 0$, we find $c = \frac{b}{\omega}$. Then,

$$x(t) = \frac{b}{\omega}(1 - \cos\omega t).$$

Summary: $\ddot{x} = \omega \frac{E}{B} - \omega \dot{z}$ $\quad \ddot{z} = \omega \dot{x}$ $\quad x(t) = \frac{b}{\omega}(1 - \cos \omega t)$

$$\dot{x}(t) = b \sin \omega t$$

$$\ddot{z} = \omega \dot{x} = \omega b \sin \omega t$$

$$\dot{z} = -b \cos \omega t + d$$

Since $\dot{z}(0) = -b + d = 0$ from the initial condition, $d = b$ and our equation becomes

$$\dot{z} = -b \cos \omega t + b \quad \text{or} \quad \dot{z} = b(1 - \cos \omega t).$$

Integrating again leads to

$$z = -\frac{b}{\omega} \sin \omega t + bt + b_0,$$ where b_0 is a constant.

The initial condition $z(0) = b_0 = 0$ means b_0 is gone. Then we have

$$z = -\frac{b}{\omega} \sin \omega t + bt \quad \text{and} \quad z = \frac{b}{\omega}(-\sin \omega t + \omega t).$$

But where did this b constant creep in. We have to get rid of that. All final answers must be in terms of the given parameters. We need to work with one of these.

$$\ddot{x} = \omega \frac{E}{B} - \omega \dot{z} \qquad \ddot{z} = \omega \dot{x}$$

PS1 (Practice Problem). Why does the second equation not work for us?

We will work with

$$\ddot{x} = \omega \frac{E}{B} - \omega \dot{z}, \quad z = \frac{b}{\omega}(\omega t - \sin \omega t), \quad \text{and} \quad x(t) = \frac{b}{\omega}(1 - \cos \omega t).$$

Substitute $\dot{z} = b(1 - \cos \omega t)$ in $\ddot{x} = \omega \frac{E}{B} - \omega \dot{z}$ to find

$$\ddot{x} = \omega \frac{E}{B} - \omega \dot{z} = \omega \frac{E}{B} - \omega b(1 - \cos \omega t)$$

Then set this equal to $\ddot{x} = \frac{d^2 x(t)}{dt^2} = \frac{d^2}{dt^2}\left[\frac{b}{\omega}(1 - \cos \omega t)\right] = b\omega \cos \omega t$.

The result is

$$b\omega \cos \omega t = \omega \frac{E}{B} - \omega b(1 - \cos \omega t).$$

The constant terms on the right side must cancel: $\omega \frac{E}{B} - \omega b = 0$. Note that the cosine terms on each side of the equation balance and we learn nothing from that. The constant b comes from the constant terms:

$$b = \frac{E}{B}.$$

The solutions $x(t) = \frac{b}{\omega}(1 - \cos \omega t)$ and $z(t) = \frac{b}{\omega}(\omega t - \sin \omega t)$ become

$$x(t) = \frac{E}{\omega B}(1 - \cos \omega t) \quad \text{and} \quad z(t) = \frac{E}{\omega B}(\omega t - \sin \omega t).$$

Let $R = \frac{E}{\omega B}$. Then $x(t) = R(1 - \cos \omega t)$ and $z(t) = R(\omega t - \sin \omega t)$. Now watch.

$$x = R - R\cos \omega t \qquad z = R\omega t - R\sin \omega t$$

$$R\cos \omega t = R - x \qquad R\sin \omega t = R\omega t - z$$

$$R^2 \cos^2 \omega t + R^2 \sin^2 \omega t = (R - x)^2 + (R\omega t - z)^2$$

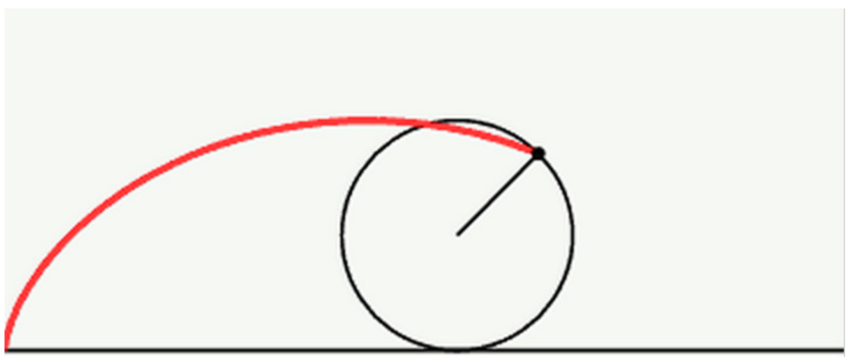
$$R^2(\cos^2\omega t + \sin^2\omega t) = (x-R)^2 + (z-R\omega t)^2$$

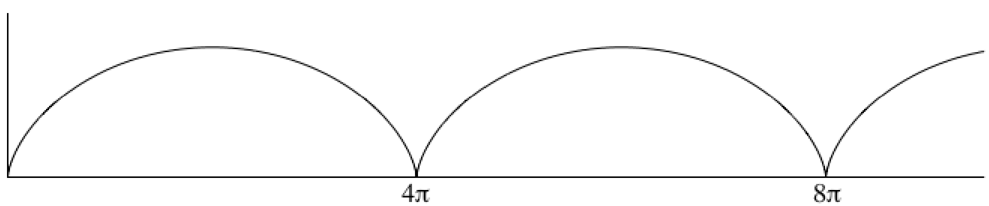
$$R^2 = (x-R)^2 + (z-R\omega t)^2$$

We can use our shift rule to figure out this is a circle whose center is at

$$(x, y, x) = (R, 0, R\omega t).$$

Note that the center moves along the z-axis. The charge q follows along the perimeter of the moving circle according to $x(t) = R(1-\cos\omega t)$ and $z(t) = R(\omega t - \sin\omega t)$. The path is a cycloid.





Courtesy Wikipedia

S3. The Vector Potential

We are in search for a potential for the magnetic field. After all, the electric field has one. In electrostatics, we have

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}, \quad \nabla \times \vec{E} = 0, \quad \text{and} \quad \vec{E} = -\nabla V.$$

In magnetostatics, we have

$$\nabla \cdot \vec{B} = 0 \quad \text{and} \quad \nabla \times \vec{B} = \mu_0 \vec{J}.$$

Note that $\nabla \times \vec{E} = 0$ means that the electric field is a gradient of a scalar: $\vec{E} = \nabla f$. Here is why.

$$\nabla \times \vec{E} = \nabla \times \nabla f = \varepsilon_{ijk} \frac{\partial}{\partial x_i} \frac{\partial f}{\partial x_j} \hat{e}_k = \varepsilon_{ijk} \frac{\partial^2 f}{\partial x_i \partial x_j} \hat{e}_k = 0$$

You get zero because the ε_{ijk} is antisymmetric in i and j, but you have symmetry in i and j with the partial derivatives which can be taken in any order. The result is zip!

PS2 (Practice). Work this out the long way without Einstein's summation convention.

Note that $\nabla \cdot \vec{B} = 0$ means that the magnetic field is a curl of a vector: $\vec{B} = \nabla \times \vec{A}$. Here is why.

$$\nabla \cdot \vec{B} = \nabla \cdot (\nabla \times \vec{A}) = \frac{\partial (\nabla \times \vec{A})_i}{\partial x_i} = \frac{\partial}{\partial x_i} (\varepsilon_{jki} \frac{\partial A_k}{\partial x_j}) = \varepsilon_{ijk} \frac{\partial^2 A_k}{\partial x_i \partial x_j} = 0$$

I used $\varepsilon_{jki} = \varepsilon_{ijk}$ since you can always make a cyclic change. Then it is easier to see that you get zero because the ε_{ijk} is antisymmetric in i and j while the derivative are symmetric in i and j. This is essentially the same argument as before.

PS3 (Practice). Work this out the long way without Einstein's summation convention.

Our summary is

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}, \quad \nabla \times \vec{E} = 0, \quad \text{and} \quad \vec{E} = -\nabla V,$$

$$\nabla \times \vec{B} = \mu_0 \vec{J}, \quad \nabla \cdot \vec{B} = 0, \quad \text{and} \quad \vec{B} = \nabla \times \vec{A}.$$