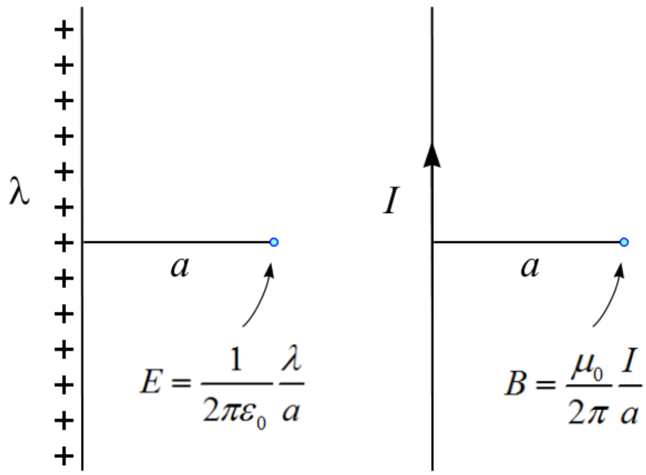
Electromagnetic Theory (EMT)
Prof. Ruiz, UNC Asheville, doctorphys on YouTube
Chapter T Notes. The Biot-Savart Law

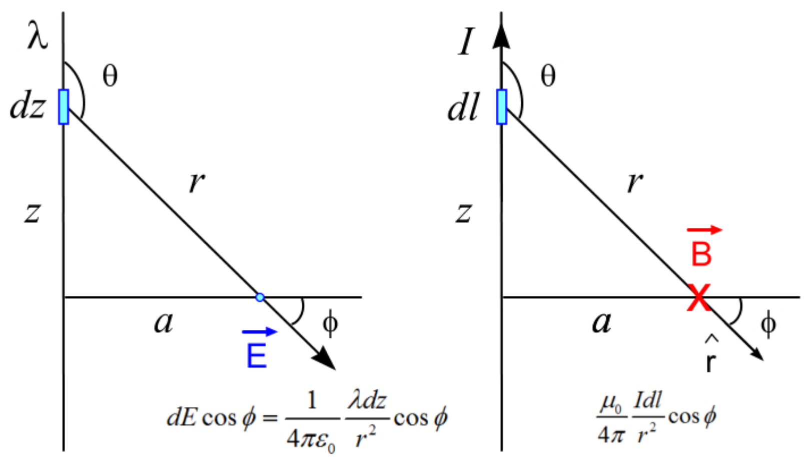
T1. Inverse-Square Law for Magnetism



Compare the magnitude of the electric field at a distance a away from an infinite line of charge to the magnitude of the magnetic field at a distance a away from an infinite line of current. The formulas are very similar.

You see the $\dfrac{1}{a}$ behavior, a source parameter λ or I, a constant, and even the 2π.

This immediately suggests an inverse-square law for a "piece" of current along the line.



We did the integration for the electric field earlier in order to calculate the result for an infinite line of charge. Of course, Gauss's Law is quicker.

Note the inverse-square law for the charge element a distance r away and the need for the cosine to obtain the component that sums up. Note: *dl* = *dz*.

There is an important difference. The magnetic field always points into the page. However, it is weaker as the angle approaches 90°. We know that the cosine term has to be there via our analogy in order to get the right integrated result. So we write

$$dB = \frac{\mu_0}{4\pi}\frac{Idl}{r^2}\cos\phi = \frac{\mu_0}{4\pi}\frac{Idl}{r^2}\sin\theta \quad \text{and} \quad \overrightarrow{dB} = \frac{\mu_0}{4\pi}\frac{I}{r^2}\overrightarrow{dl}\times\hat{r}$$

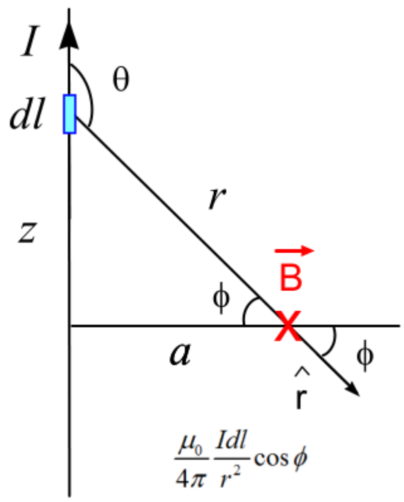
PT1 (Practice Problem). Show that $\cos\phi = \sin\theta$.

T2. The Biot-Savart Law

Our formula for the magnetic field due to an element of current is known as the Biot-Savart Law. We can write it as a differential or an integral.

$$\overrightarrow{dB} = \frac{\mu_0}{4\pi}\frac{I}{r^2}\overrightarrow{dl}\times\hat{r} \qquad \vec{B} = \frac{\mu_0}{4\pi}\int\frac{I}{r^2}\overrightarrow{dl}\times\hat{r}$$

Let's integrate for the infinite line of current to double check things.



$$B = \frac{\mu_0}{4\pi}\int\frac{I}{r^2}\sin\theta dl = \frac{\mu_0}{4\pi}\int\frac{I}{r^2}\cos\phi dl$$

$$\cos\phi = \frac{a}{r}$$

$$B = \frac{\mu_0}{4\pi}\int\frac{Ia}{r^3}dz$$

We will use $\tan\phi = \frac{z}{a}$, i.e., $z = a\tan\phi$.

PT2 (Practice Problem). Show that $dz = \frac{a}{\cos^2\phi}d\phi$ from $\frac{d}{d\theta}\left[\frac{\sin\theta}{\cos\theta}\right]$.

$$B = \frac{\mu_0}{4\pi}\int\frac{Ia}{r^3}dz = \frac{\mu_0 Ia}{4\pi}\int\frac{1}{(a^2+z^2)^{3/2}}dz$$

PT3 (Practice Problem). We did this integral in Chapter C. Use that result to finish it.

We will do it another way.

$$B = \frac{\mu_0 Ia}{4\pi}\int_{-\pi/2}^{\pi/2}\frac{1}{(a^2+a^2\tan^2\theta)^{3/2}}\frac{a}{\cos^2\phi}d\phi$$

PT4 (Practice Problem). Show that $1+\tan^2\phi = \frac{1}{\cos^2\phi}$.

First factor out the a^2 as a^3 in the denominator due to the 3/2 power.

$$B = \frac{\mu_0 I a}{4\pi a^3}\int_{-\pi/2}^{\pi/2}\frac{1}{(1+\tan^2\theta)^{3/2}}\frac{a}{\cos^2\phi}d\phi$$

Now use

$$1+\tan^2\phi = \frac{1}{\cos^2\phi} \qquad \text{to get}$$

$$B = \frac{\mu_0 I a}{4\pi a^3}\int_{-\pi/2}^{\pi/2}\frac{1}{(1/\cos^3\phi)}\frac{a}{\cos^2\phi}d\phi \; .$$

$$B = \frac{\mu_0 I}{4\pi a}\int_{-\pi/2}^{\pi/2}\frac{\cos^3\phi}{\cos^2\phi}d\phi$$

Well, we have the correct factor $1/a$, which is nice.

Remember that we know the answer: $B = \frac{\mu_0 I}{2\pi a}$. We are just double checking here.

$$B = \frac{\mu_0 I}{4\pi a}\int_{-\pi/2}^{\pi/2}\cos\phi\, d\phi$$

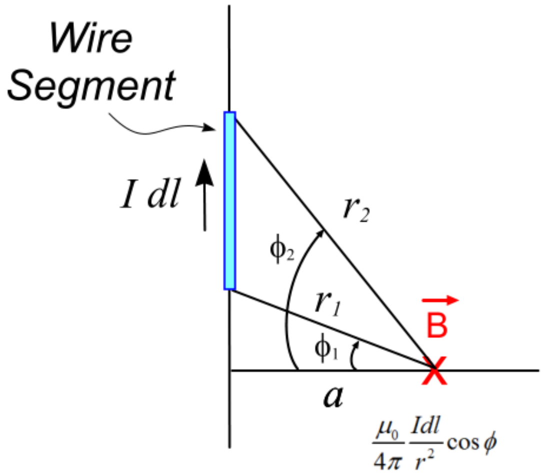
Since the cosine is an even function and we have a symmetric range, we can write

$$B = \frac{\mu_0 I}{2\pi a}\int_{0}^{\pi/2}\cos\phi\, d\phi \; .$$

The integral better evaluate to 1.

$$B = \frac{\mu_0 I}{2\pi a}\sin\phi\Big|_0^{\pi/2} = \frac{\mu_0 I}{2\pi a}(\sin\frac{\pi}{2} - \sin 0) = \frac{\mu_0 I}{2\pi a}(1-0)$$

$$B = \frac{\mu_0 I}{2\pi a} \qquad \vec{B} = \frac{\mu_0 I}{2\pi a}\hat{\theta}$$



The Finite Wire. We can now easily do what initially might appear as a difficult problem.

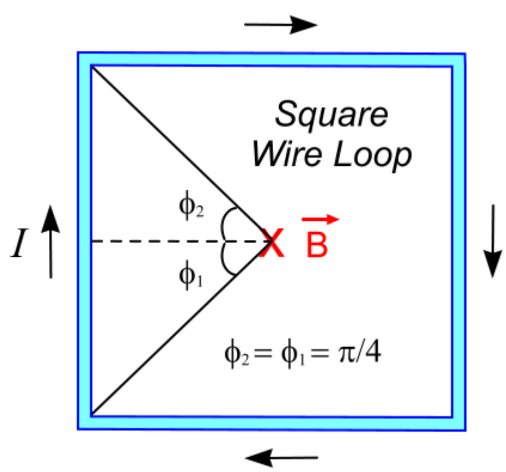
$$B = \frac{\mu_0 I}{4\pi a}\int_{\phi_1}^{\phi_2}\cos\phi\, d\phi$$

Integrate from some initial angle to some final one, both smaller than 90° and we have a more general solution.

$$B = \frac{\mu_0 I}{4\pi a}\sin\phi\Big|_{\phi_1}^{\phi_2}$$

$$B = \frac{\mu_0 I}{4\pi a}(\sin\phi_2 - \sin\phi_1)$$

The Square Wire. From this we can do an even scarier problem: the magnetic field at the center of a square wire loop.



We apply the finite equation on the left vertical wire where we multiply by 4.

$$B = 4\frac{\mu_0 I}{4\pi a}(\sin\phi_2 - \sin\phi_1)$$

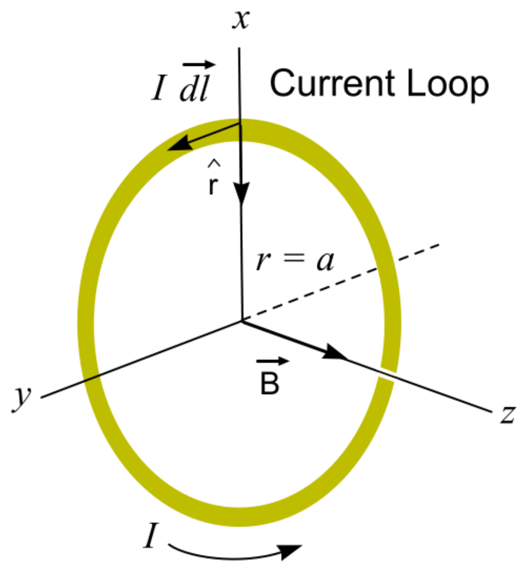
$$B = \frac{\mu_0 I}{\pi a}\left[\sin\frac{\pi}{4} - \sin(-\frac{\pi}{4})\right]$$

$$B = \frac{\mu_0 I}{\pi a}\left[\sin\frac{\pi}{4} + \sin\frac{\pi}{4}\right]$$

$$B = \frac{\mu_0 I}{\pi a}2\sin\frac{\pi}{4} = \frac{\mu_0 I}{\pi a}2\frac{\sqrt{2}}{2}$$

$$B = \sqrt{2}\frac{\mu_0 I}{\pi a}$$

T3. Ring of Current



A nice basic problem is to calculate the magnetic field at the center of a circular loop of current in the x-y plane. We start with the Biot-Savart Law.

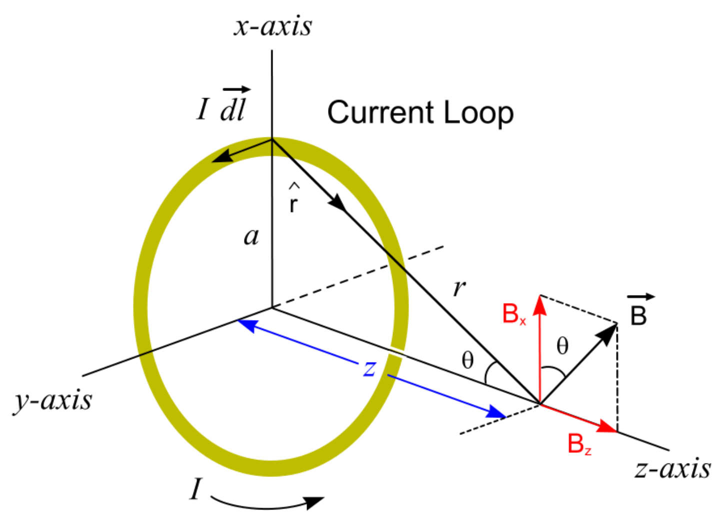
$$\vec{B} = \frac{\mu_0}{4\pi}\int \frac{I}{r^2}\vec{dl}\times\hat{r}$$

$$\vec{dl}\times\hat{r} = \hat{k}\,dl$$

This is true for any $\vec{dl}$ along the loop. The magnetic field points along the z-axis.

$$\vec{B} = B\,\hat{k}$$

$$B = \frac{\mu_0}{4\pi}\int_0^{2\pi a}\frac{I}{r^2}dl = \frac{\mu_0}{4\pi}\frac{I}{a^2}\int_0^{2\pi a}dl = \frac{\mu_0}{4\pi}\frac{I}{a^2}2\pi a = \frac{\mu_0 I}{2a}$$



Now we proceed to a point along the z-axis. Note that the cross product still includes an angle of 90°, but now the magnetic field is no longer along the z-axis.

However, we want B_z since the x-component will cancel out. The best news is that the magnitude of the magnetic field is constant, which means the integral will be trivial. Watch!

$$B_z = \frac{\mu_0}{4\pi}\int_0^{2\pi a}\frac{I}{r^2}dl\sin\theta = \frac{\mu_0}{4\pi}\int_0^{2\pi a}\frac{I}{(z^2+a^2)}dl\frac{a}{\sqrt{z^2+a^2}}$$

$$B_z = \frac{\mu_0}{4\pi}\frac{Ia}{(z^2+a^2)^{3/2}}\int_0^{2\pi a}dl = \frac{\mu_0}{4\pi}\frac{Ia(2\pi a)}{(z^2+a^2)^{3/2}} = \frac{\mu_0}{2\pi}\frac{I\pi a^2}{(z^2+a^2)^{3/2}}$$

T4. Solenoids

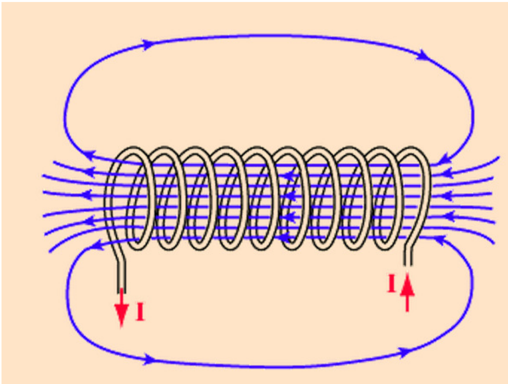


Figure Courtesy HyperPhysics, R Nave

A solenoid is a circuit element with wire windings. See the figure at the left for the windings, current, and a sketch of magnetic field lines.

Infinitely Long Solenoid. For cases of symmetry we use Ampère's Law instead of the Biot-Savart Law. We will do this below for the infinitely long solenoid.

The windings per unit length is n, sometimes called the turn density.

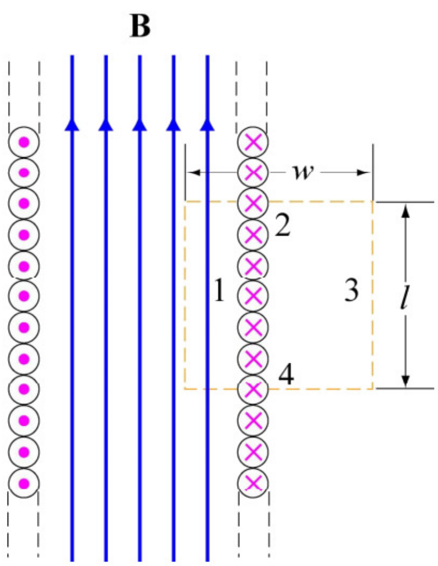


Figure Courtesy MIT Open Courseware

$$\oint \vec{B} \cdot \vec{dl} = \mu_0 I_{enc}$$

Refer to the figure for the Ampèrian path and path components 1 through 4.

$$\oint \vec{B} \cdot \vec{dl} = \int_1 Bdl + \int_2 Bdl + \int_3 Bdl + \int_4 Bdl$$

$$\oint \vec{B} \cdot \vec{dl} = Bl + 0 + 0 + 0$$

$I_{enc} = nlI$, where n is the turn density.

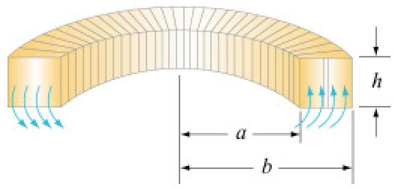
Then, $\oint \vec{B} \cdot \vec{dl} = \mu_0 I_{enc}$ leads to $Bl = \mu_0 nlI$ and $B = \mu_0 nI$.

The self inductance L is defined as $\Phi_B = LI$, where Φ_B is the magnetic flux.

Then, the voltage $\mathcal{E} = \oint \vec{E} \cdot \vec{dl} = -\frac{d\Phi_B}{dt}$ leads to $\mathcal{E} = \oint \vec{E} \cdot \vec{dl} = -L\frac{dI}{dt}$.

For our solenoid section l: $L_l = \frac{\Phi_B}{I} = \frac{nlBA}{I} = \frac{nl(\mu_0 nI)A}{I} = \mu_0 n^2 Al$.

Figure Courtesy MIT Open Courseware



The Toroidal Solenoid. See the figure at the left to the geometry of the toroidal solenoid. Take the Ampèrian path to be a circle inside the toroid where $a < r < b$.

$$\oint \vec{B} \cdot \vec{dl} = \mu_0 I_{enc}$$

$$B 2\pi r = \mu_0 NI \qquad B 2\pi r = \mu_0 NI \qquad B = \frac{\mu_0 NI}{2\pi r}$$

PT5 (Practice Problem). Show that the toroidal inductance is $L = \frac{\mu_0 N^2 h}{2\pi} \ln \frac{b}{a}$.

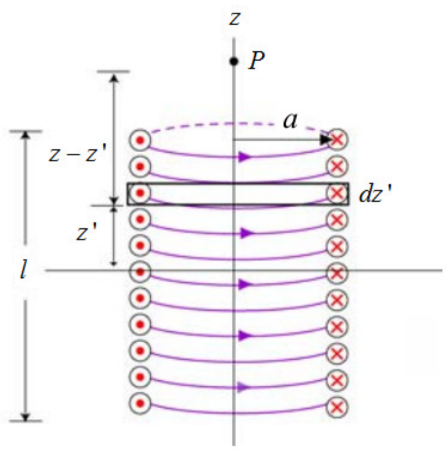


Figure Courtesy MIT Open Courseware

The Finite Solenoid. We consider a tightly wound finite linear solenoid with N windings over a distance of l. We will determine the magnetic field along the axis. The number of windings per length is

$$n = \frac{N}{l}.$$

We use our result from before for the magnetic field along the z-axis from a circle or current. Our formula from before,

$$\vec{B}(z) = \frac{\mu_0}{2\pi} \frac{I\pi a^2}{(z^2 + a^2)^{3/2}} \hat{k},$$ gets modified to

$$\overrightarrow{dB}(z - z') = \frac{\mu_0}{2\pi} \frac{dI \pi a^2}{\left[(z - z')^2 + a^2\right]^{3/2}} \hat{k}$$ with $dI = (I)(ndz')$.

$$B_z = \frac{\mu_0}{2\pi} \int_{-l/2}^{l/2} \frac{\pi a^2 n I dz'}{\left[(z - z')^2 + a^2\right]^{3/2}}$$

$$B_z = \frac{\mu_0 n I a^2}{2} \int_{-l/2}^{l/2} \frac{dz'}{\left[(z - z')^2 + a^2\right]^{3/2}}$$

We can do this integral using our integral friend from earilier.

$$I = \int \frac{dx}{(x^2 + a^2)^{3/2}} = \frac{1}{a^2} \frac{x}{\sqrt{x^2 + a^2}}$$

Let $x = z' - z$, where z is a constant. Then $dx = dz'$.

$$I = \int \frac{dz'}{\left[(z' - z)^2 + a^2\right]^{3/2}} = \frac{1}{a^2} \frac{z' - z}{\sqrt{(z' - z)^2 + a^2}}$$

$$B_z = \frac{\mu_0 n I a^2}{2} \int_{-l/2}^{l/2} \frac{dz'}{\left[(z - z')^2 + a^2\right]^{3/2}}$$

$$B_z = \frac{\mu_0 n I a^2}{2} \frac{1}{a^2} \frac{z' - z}{\sqrt{(z' - z)^2 + a^2}} \Bigg|_{-l/2}^{l/2}$$

$$B_z = \frac{\mu_0 n I}{2} \left[\frac{(l/2) - z}{\sqrt{(\frac{l}{2} - z)^2 + a^2}} - \frac{(-l/2) - z}{\sqrt{(-\frac{l}{2} - z)^2 + a^2}} \right]$$

$$B_z = \frac{\mu_0 n I}{2} \left[\frac{(l/2) - z}{\sqrt{(z - l/2)^2 + a^2}} + \frac{(l/2) - z}{\sqrt{(z + l/2)^2 + a^2}} \right]$$

PT6 (Practice Problem). Obtain the infinite solenoid result from this formula.

PT7 (Practice Problem). Find $B_z(0)$. What is the first correction term in the expansion for small radius a? Your leading term should be the infinite solenoid result.