Electromagnetic Theory (EMT)
Prof. Ruiz, UNC Asheville, doctorphys on YouTube
Chapter U Notes. Energy

U1. Energy in an Electric Field

Bring a charge q_2 to a distance r away from q_1. Consider the two charges positive and q_1 pasted at the origin. You have to apply an inward force to push the q_2 in since these charges repel. So you have to push against the outward force $q_2\vec{E}$.

$$W = \int_{\infty}^{r} \vec{F}_{applied} \cdot \vec{dl} = -q_2 \int_{\infty}^{r} \vec{E} \cdot \vec{dl} = +q_2 V(r)\Big|_{\infty}^{r} = q_2\left[V(r) - V(\infty)\right]$$

We take the potential at infinity as our zero reference.

$$W = q_2 V(r)$$

The potential $V(r)$ is due to q_1: $V(r) = \frac{1}{4\pi\varepsilon_0}\frac{q_1}{r}$.

To emphasize that the distance r is between the two charges we write

$$V(r) = \frac{1}{4\pi\varepsilon_0}\frac{q_1}{r_{12}}.$$

The work is $W = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r_{12}}$.

Now we bring in a third charge to the two charges q_1 and q_2 already in place.

$W_{extra} = q_3 V_{new}(r)$, where $W_{extra} = \frac{1}{4\pi\varepsilon_0}\left[\frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}}\right]$.

The total work so far is $W = \frac{1}{4\pi\varepsilon_0}\left[\frac{q_1 q_2}{r_{12}} + \frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}}\right]$.

The general formula for n charges is $W = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\sum_{\substack{j=1\\j>i}}^{n}\frac{q_i q_j}{r_{ij}}$.

$$W = \frac{1}{4\pi\varepsilon_0}\frac{1}{2}\sum_{i=1}^{n}\sum_{\substack{j=1\\j\neq i}}^{n}\frac{q_i q_j}{r_{ij}}$$

$$W = \frac{1}{2}\sum_{i=1}^{n} q_i\left[\sum_{\substack{j=1\\j\neq i}}^{n}\frac{1}{4\pi\varepsilon_0}\frac{q_j}{r_{ij}}\right]$$

$$W = \frac{1}{2}\sum_{i=1}^{n} q_i V(\vec{r}_i)$$

$$W = \frac{1}{2}\int \rho(\vec{r})V(\vec{r})d\tau$$

$$\nabla\cdot\vec{E} = \frac{\rho}{\varepsilon_0}$$

$$W = \frac{\varepsilon_0}{2}\iiint_\tau (\nabla\cdot\vec{E})V(\vec{r})d\tau = \frac{\varepsilon_0}{2}\iiint_\tau V(\nabla\cdot\vec{E})d\tau$$

$$\nabla\cdot(f\vec{A}) = \vec{A}\cdot\nabla f + f\nabla\cdot\vec{A}$$

$$f\nabla\cdot\vec{A} = \nabla\cdot(f\vec{A}) - \vec{A}\cdot\nabla f$$

$$V\nabla\cdot\vec{E} = \nabla\cdot(V\vec{E}) - \vec{E}\cdot\nabla V$$

$$W = \frac{\varepsilon_0}{2}\iiint_\tau \nabla\cdot(V\vec{E})d\tau - \frac{\varepsilon_0}{2}\iiint_\tau \vec{E}\cdot\nabla V d\tau$$

$$W = \frac{\varepsilon_0}{2} \iiint_\tau \nabla \cdot (V\vec{E}) d\tau - \frac{\varepsilon_0}{2} \iiint_\tau \vec{E} \cdot (-\vec{E}) d\tau$$

$$W = \frac{\varepsilon_0}{2} \iiint_\tau \nabla \cdot (V\vec{E}) d\tau + \frac{\varepsilon_0}{2} \iiint_\tau \vec{E} \cdot \vec{E} d\tau$$

Use the Divergence Theorem $\oiint \vec{F} \cdot \vec{da} = \iiint_\tau \nabla \cdot \vec{F} d\tau$ on the first term.

$$\boxed{W = \frac{\varepsilon_0}{2} \oiint V\vec{E} \cdot \vec{da} + \frac{\varepsilon_0}{2} \iiint_\tau E^2 d\tau}$$

Take the volume to be large as any volume will do as long you enclose the charges. But the potential is zero at infinity. So for the super large volume the first integral vanishes.

Here is what is left. $W = \frac{\varepsilon_0}{2} \iiint_{all\ space} \vec{E} \cdot \vec{E} d\tau$ $\boxed{W = \frac{\varepsilon_0}{2} \iiint_{all\ space} E^2 d\tau}$

Energy per unit volume is given the the next equation. $\boxed{u = \frac{1}{2} \varepsilon_0 E^2}$

PU1 (Practice Problem). From this equation, justify that the energy stored in a parallel-plate capacitor is

$$W = \frac{\varepsilon_0}{2} E^2 Ad \; .$$

U2. Energy in a Magnetic Field

Faraday's Law in Integral Form: $\mathcal{E} = \oint \vec{E} \cdot \vec{dl} = -\frac{d\Phi_B}{dt} = -L\frac{dI}{dt}$

Power for the Current: $P = IV$

To fight the back EMF so we can produce the current, we apply

$$P_{applied} = \frac{dW_{applied}}{dt} = -I\mathcal{E} = LI\frac{dI}{dt} \quad \text{giving} \quad W_{applied} = \frac{1}{2}LI^2 .$$

$$W = \frac{1}{2}LI^2$$

Faraday's Law in Differential Form (a Maxwell Equation): $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

Magnetic Field from Vector Potential: $\vec{B} = \nabla \times \vec{A}$

$$\nabla \times \vec{E} = -\frac{\partial}{\partial t}(\nabla \times \vec{A})$$

$$\nabla \times \vec{E} = -\nabla \times \frac{\partial \vec{A}}{\partial t}$$

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t}$$

But don't forget the basic production of electric fields from static charges:

$\vec{E} = -\nabla V$ So in general $\vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t}$.

For electric fields generated by magnetic flux changes we just need $\vec{E} = -\frac{\partial \vec{A}}{\partial t}$.

$\mathcal{E} = \oint \vec{E} \cdot \vec{dl} = -\frac{d\Phi_B}{dt}$ can be expressed as $\mathcal{E} = -\frac{d}{dt}\oint \vec{A} \cdot \vec{dl}$

$\mathcal{E} = -\frac{d\Phi_B}{dt} = -L\frac{dI}{dt}$ leads to $\frac{d}{dt}\oint \vec{A} \cdot \vec{dl} = L\frac{dI}{dt}$ and $\oint \vec{A} \cdot \vec{dl} = LI$

Summary: $W = \frac{1}{2}LI^2$ and $\oint \vec{A} \cdot \vec{dl} = LI$.

$$W = \frac{1}{2} I \oint \vec{A} \cdot \overrightarrow{dl}$$

$$I = \iint \vec{J} \cdot \overrightarrow{da} = \iint \vec{J} \cdot \hat{n}\, da$$

The current direction is normal to the cross section: $\overrightarrow{dl} = l\,\hat{\mathrm{n}}$.

$$W = \frac{1}{2} I \oint \vec{A} \cdot \hat{n}\, dl$$

$$W = \frac{1}{2} \iint \vec{J} \cdot \hat{n}\, da \oint \vec{A} \cdot \hat{n}\, dl$$

$$W = \frac{1}{2} \iiint \vec{J} \cdot \vec{A} dadl$$

$$W = \frac{1}{2} \iiint_{\tau} \vec{J} \cdot \vec{A} d\tau$$ or $$W = \frac{1}{2} \iiint_{\tau} \vec{A} \cdot \vec{J} d\tau$$

$$\nabla \times \vec{B} = \mu_0 \vec{J}$$

$$W = \frac{1}{2\mu_0} \iiint_{\tau} \vec{A} \cdot (\nabla \times \vec{B}) d\tau$$

$$\nabla \cdot (\vec{A} \times \vec{B}) = \frac{\partial}{\partial x_n} \hat{e}_n \cdot (\varepsilon_{ijk} A_i B_j \hat{e}_k) = \delta_{nk} \frac{\partial}{\partial x_n} (\varepsilon_{ijk} A_i B_j) = \frac{\partial}{\partial x_k} (\varepsilon_{ijk} A_i B_j)$$

$$\nabla \cdot (\vec{A} \times \vec{B}) = \frac{\partial}{\partial x_k} (\varepsilon_{ijk} A_i B_j) = \varepsilon_{ijk} \frac{\partial A_i}{\partial x_k} B_j + \varepsilon_{ijk} A_i \frac{\partial B_j}{\partial x_k}$$

$$\nabla \cdot (\vec{A} \times \vec{B}) = \varepsilon_{kij} \frac{\partial A_i}{\partial x_k} B_j + \varepsilon_{kij} A_i \frac{\partial B_j}{\partial x_k}$$

$$\nabla \cdot (\vec{A} \times \vec{B}) = \varepsilon_{kij} \frac{\partial A_i}{\partial x_k} B_j - \varepsilon_{kji} A_i \frac{\partial B_j}{\partial x_k}$$

$$\nabla \cdot (\vec{A} \times \vec{B}) = (\nabla \times \vec{A}) \cdot \vec{B} - \vec{A} \cdot (\nabla \times \vec{B})$$

$$\nabla \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\nabla \times \vec{A}) - \vec{A} \cdot (\nabla \times \vec{B})$$

$$W = \frac{1}{2\mu_0} \iiint_\tau \vec{A} \cdot (\nabla \times \vec{B}) d\tau$$

$$\vec{A} \cdot (\nabla \times \vec{B}) = \vec{B} \cdot (\nabla \times \vec{A}) - \nabla \cdot (\vec{A} \times \vec{B})$$

$$\vec{A} \cdot (\nabla \times \vec{B}) = \vec{B} \cdot \vec{B} - \nabla \cdot (\vec{A} \times \vec{B})$$

$$W = \frac{1}{2\mu_0} \iiint_\tau \vec{B} \cdot \vec{B} d\tau - \frac{1}{2\mu_0} \iiint_\tau \nabla \cdot (\vec{A} \times \vec{B}) d\tau$$

$$W = \frac{1}{2\mu_0} \iiint_\tau \vec{B} \cdot \vec{B} d\tau - \frac{1}{2\mu_0} \oiint (\vec{A} \times \vec{B}) \cdot \overrightarrow{da}$$

Large volume gives decreasing vector potential and magnetic fields.

$$W = \frac{1}{2\mu_0} \iiint_\tau \vec{B} \cdot \vec{B} d\tau \qquad \boxed{W = \frac{1}{2\mu_0} \iiint_\tau B^2 d\tau}$$

Energy per unit volume: $\boxed{u = \frac{1}{2\mu_0} B^2}$

PU2 (Practice Problem). From this equation, justify that the energy stored in a coil with current is $W = \frac{1}{2\mu_0} B^2 A l$.

U3. Energy in an Electromagnetic Wave

Summary from an earlier Chapter.

$$\nabla^2 \vec{E} = \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2}$$

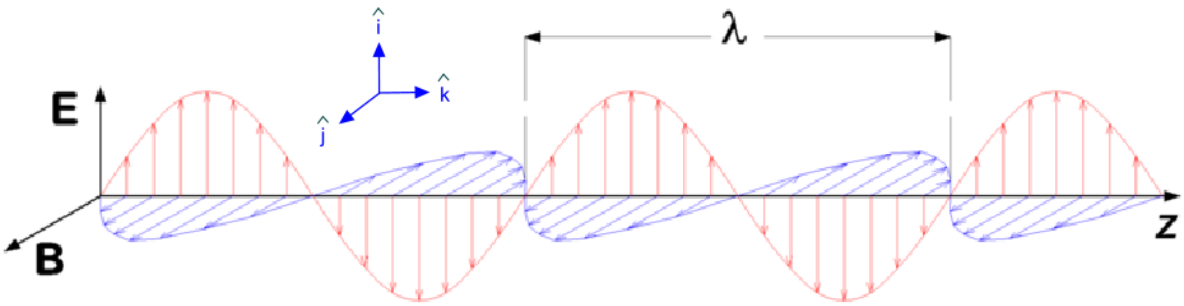
$$\nabla^2 \vec{B} = \frac{1}{c^2}\frac{\partial^2 \vec{B}}{\partial t^2}$$

$$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = \lambda f$$

$$\vec{E} = E_0 \sin\left[k(z-ct)\right]\hat{i}$$

$$\vec{B} = B_0 \sin\left[k(z-ct)\right]\hat{j}$$

$$k = \frac{2\pi}{\lambda}$$



EM Wave Courtesy P.wormer, Wikimedia

Faraday's Law in Differential Form (a Maxwell Equation): $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{E} = E_0 \sin\left[k(z-ct)\right]\hat{i}$$

$$\nabla \times \vec{E} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_0 \sin k(z-ct) & 0 & 0 \end{vmatrix} = -\hat{j}\left[0 - kE_0 \cos k(x-ct)\right]$$

$$\nabla \times \vec{E} = kE_0 \cos k(x-ct)\,\hat{j}$$

$$\nabla \times \vec{E} = kE_0 \cos k(x-ct)\,\hat{j} \quad \text{and} \quad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$kE_0 \cos k(x-ct)\,\hat{j} = -\frac{\partial \vec{B}}{\partial t} = -B_0 \frac{\partial}{\partial t} \sin\left[k(z-ct)\right]\hat{j}$$

$$kE_0 \cos k(x-ct)\,\hat{j} = -B_0 \cos\left[k(z-ct)\right](-kc)\,\hat{j}$$

$$kE_0 = -B_0(-kc)$$

$$\boxed{E_0 = cB_0} \qquad \boxed{B_0 = \frac{E_0}{c}} \qquad \boxed{c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}}$$

$$\boxed{\vec{E} = E_0 \sin\left[k(z-ct)\right]\hat{i}} \qquad \boxed{\vec{B} = \frac{E_0}{c} \sin\left[k(z-ct)\right]\hat{j}}$$

Energy per unit volume: $u = \frac{\varepsilon_0}{2}E^2 + \frac{1}{2\mu_0}B^2$. $\quad B^2 = \frac{E^2}{c^2} = \mu_0 \varepsilon_0 E^2$

$$u_E = \frac{\varepsilon_0}{2}E^2 \quad \text{and} \quad u_B = \frac{1}{2\mu_0}B^2 = \frac{1}{2\mu_0}\mu_0\varepsilon_0 E^2 = \frac{\varepsilon_0}{2}E^2 = u_E$$

$$u = \varepsilon_0 E^2 = \varepsilon_0 E_0^2 \sin^2\left[k(z-ct)\right]$$

$$\vec{E} \times \vec{B} = \frac{E_0^2}{c^2} \sin^2\left[k(z-ct)\right]\hat{k}$$ Define $\vec{S} = cu\hat{k}$.

Then, $\vec{S} = \frac{1}{\mu_0}\vec{E} \times \vec{B}$. Poynting vector ("points" in wave direction).

PU3 (Practice Problem). Show that the unit for the Poynting vector is energy per unit area per unit time.