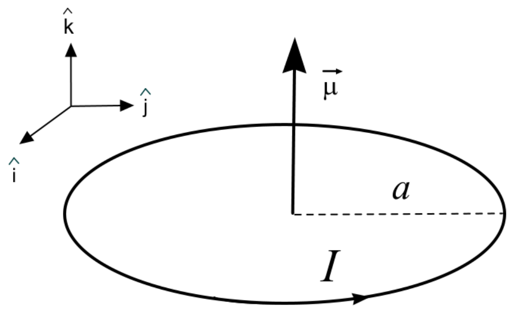
Electromagnetic Theory (EMT)
Prof. Ruiz, UNC Asheville, doctorphys on YouTube
Chapter V Notes. Inductors

V1. Magnetic Dipole Moment



1. Magnetic Dipole. The magnetic moment of a loop of current is defined as the product of the current and area with a unit vector perpendicular to the current loop.

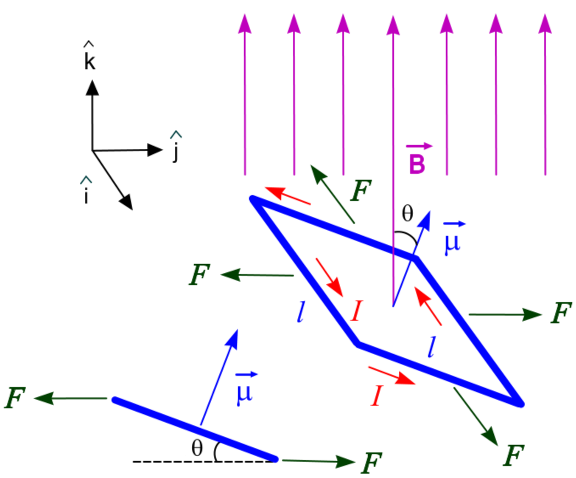
$$\vec{\mu} = IA\hat{k} = I\pi a^2 \hat{k}$$

In our next section we will need the force on a line of wire due to a magnetic field. We will derive the result we need. Let the linear charge density be λ_q .

$$\vec{F} = \int \vec{v} \times \vec{B} dq \qquad dq = \lambda_q dl \qquad \vec{F} = \int \vec{v} \times \vec{B} \lambda_q dl \qquad v\lambda_q = I$$

$$\vec{F} = \int \vec{I} \times \vec{B} dl \qquad \boxed{\vec{F} = \int I\, d\vec{l} \times \vec{B}}$$

Simplest cases: $\boxed{\vec{F} = I\vec{l} \times \vec{B}}$ $\boxed{F = IlB}$



2. Torque on a Magnetic Dipole.

Let's figure out the force on the right wire in the loop the conservative safe way using vectors.

$$\vec{F} = I\vec{l} \times \vec{B} = -Il\hat{i} \times B\hat{k}$$

$$\vec{F} = IlB\hat{k} \times \hat{i} = IlB\hat{j}$$

Using vectors is very important in engineering when things can get very complicated quickly.

PV1 (Practice Problem). Work out the other force vectors the conservative way.

From the lower left inset (above figure), the torque is $\tau = 2\dfrac{l}{2}F\sin\theta$ with $F = IlB$. Thus, $\tau = l(IlB)\sin\theta = Il^2 B\sin\theta$. Note $\vec{\tau} = \tau\hat{i}$ and $\vec{\tau} = \vec{\mu} \times \vec{B}$.

Compare the electric dipole torque in an electric field and the magnetic dipole in a magnetic field.

$$\boxed{\vec{\tau} = \vec{p} \times \vec{E}} \qquad \boxed{\vec{\tau} = \vec{\mu} \times \vec{B}}$$

We can pull twist the magnetic dipole and find the potential energy. It will have the same form as the electric case.

$$\boxed{U_E = -\vec{p} \cdot \vec{E}} \qquad \boxed{U_M = -\vec{\mu} \cdot \vec{B}}$$

V2. Magnetization

$$\varepsilon_0 \vec{E} = \vec{D} - \vec{P} \qquad \frac{1}{\mu_0} \vec{B} = \vec{H} + \vec{M}$$

The magnetization vector M is analogous to the polarization vector P. Since the induced magnetic alignment inside a medium goes with the applied magnetic field rather than against as in the case of electric fields and dielectrics, we use the plus sign in front of the magnetization vector.

$$\boxed{\vec{D} = \varepsilon_0 \vec{E} + \vec{P}} \qquad \boxed{\vec{H} = \frac{1}{\mu_0} \vec{B} - \vec{M}}$$

$$\boxed{\nabla \cdot \vec{D} = \rho_f} \qquad \boxed{\nabla \times \vec{H} = \vec{J}_f}$$

$$\boxed{\oiint \vec{D} \cdot \vec{da} = \rho_f} \qquad \boxed{\oint \vec{H} \cdot \vec{dl} = I_f}$$

$\boxed{\vec{P} \equiv \varepsilon_0 \chi_e \vec{E}}$ Not as expected, $\vec{M} \neq \frac{1}{\mu_0} \chi_m \vec{B}$, but $\boxed{\vec{M} \equiv \chi_m \vec{H}}$

Magnetic susceptibility χ_m .

Analogous equations deviate at this point due to the different definition. We now get:

$$\vec{H} = \frac{1}{\mu_0} \vec{B} - \vec{M} \qquad \mu_0 \vec{H} = \vec{B} - \mu_0 \vec{M} \qquad \boxed{\vec{B} = \mu_0 (\vec{H} + \vec{M})}$$

$$\vec{B} = \mu_0(\vec{H} + \vec{M}) \qquad \vec{M} \equiv \chi_m \vec{H}$$

$$\boxed{\vec{B} = \mu_0(1+\chi_m)\vec{H}} \qquad \boxed{\mu = \mu_0(1+\chi_m)} \qquad \boxed{\vec{B} = \mu\vec{H}}$$

The permeability is μ . The permeability of free space is μ_0 .

The Three Magnetic Field Vectors

$\vec{H}$	Applied magnetic field strength (amps/meter = A/m)
$\vec{M}$	Magnetic field strength due to dipoles in material (A/m)
$\vec{B}$	Total magnetic field (unit is the tesla)

PV1 (Practice Problem). Use the "big 4" base units to give the dimensions.

L = length, T = time, M = mass, Q = charge

Q	Q	coulomb
V	$ML^2T^{-2}Q^{-1}$	volt
I	QT^{-1}	amp = coulomb / second
R	$ML^2T^{-1}Q^{-2}$	ohm = volt / amp
C	$M^{-1}L^{-2}T^2Q^2$	farad = coulomb / volt
L	ML^2Q^{-2}	henry = volt • second / amp
Power P	ML^2T^{-3}	watt = volt • amp
E	$MLT^{-2}Q^{-1}$	volt / meter
D	$L^{-2}Q$	coulomb / meter^2
Dipole P	$L^{-2}Q$	coulomb / meter^2
B	$MT^{-1}Q^{-1}$	tesla = weber / meter^2
H	$L^{-1}T^{-1}Q$	amp / meter
M	$L^{-1}T^{-1}Q$	amp / meter
Φ_B	$ML^2T^{-1}Q^{-1}$	weber = volt • second

The external applied field is $\vec{H}$.

$$\vec{M} = \chi_m \vec{H} \qquad \vec{B} = \mu_0 (1 + \chi_m) \vec{H}$$

χ_m large and positive: ferromagnetic - applied H field greatly strengthened by M

χ_m small and positive: paramagnetic - applied H field is strengthened by M

χ_m small and negative: diamagnetic - applied H field is weekend by M

Ferromagnetism. Ferromagnetism refers to materials that have a high magnetic susceptibility. They can keep an internal magnetic alignment of little dipoles without needing an external magnetic field. Ferromagnetic Materials are non-linear. This leads to the hysteresis loop, a curve that shows reversing an external magnetic field does not bring the material back along is original magnetized path.

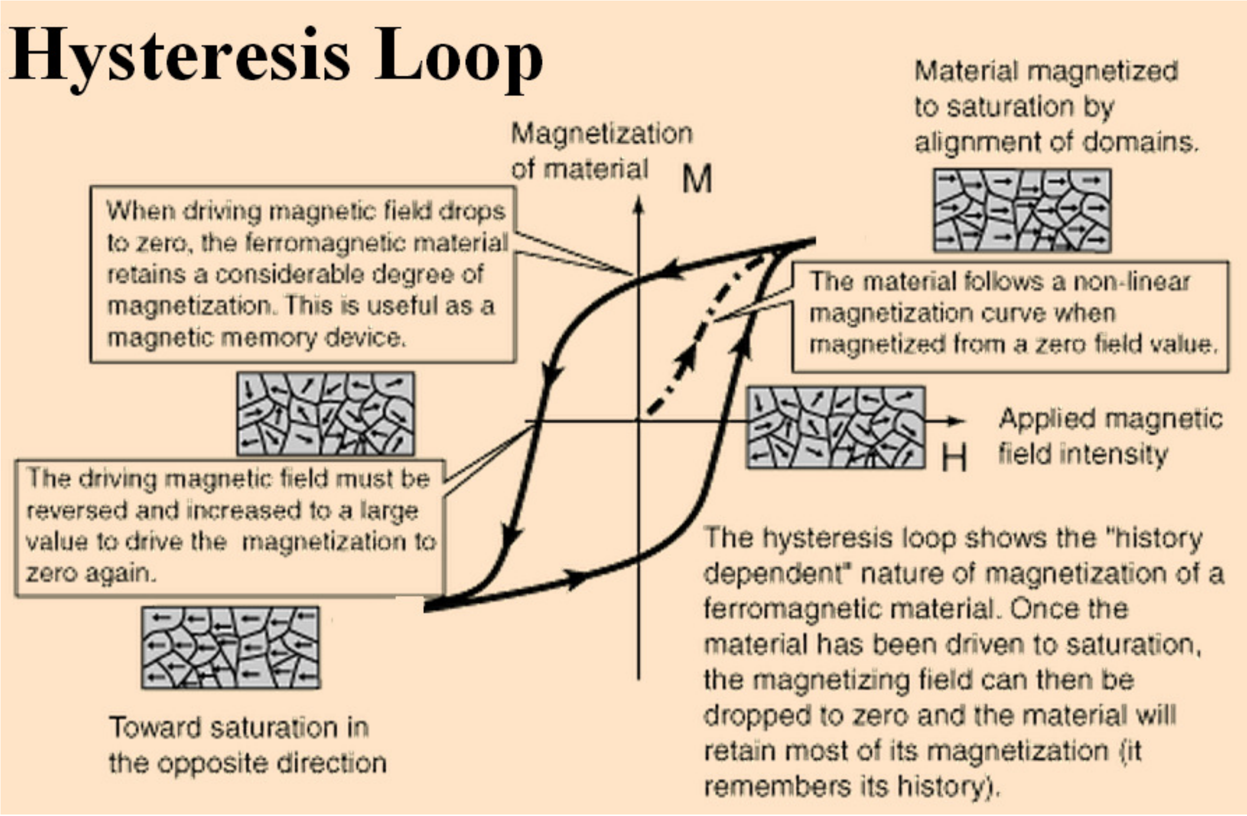


Figure Courtesy HyperPhysics, R. Nave
Reference: Young, Hugh D., *University Physics*, 8th ed., Addison-Wesley, 1992.

V3. The Vector Potential and Current Density

Let's pause to relate the vector potential to the current density.

$$\nabla \times \vec{B} = \mu_0 \vec{J} \qquad \vec{B} = \nabla \times \vec{A}$$

$$\nabla \times \vec{B} = \nabla \times (\nabla \times \vec{A})$$

We already calculated $\nabla \times (\nabla \times \vec{A})$ when we solved the Maxwell equations in free space to derive the wave equation. We found

$$\nabla \times (\nabla \times \vec{E}) = \nabla(\nabla \cdot \vec{E}) - \nabla^2 \vec{E}.$$

Then since $\nabla \cdot \vec{E} = 0$ in free space, we obtained $\nabla \times (\nabla \times \vec{E}) = -\nabla^2 \vec{E}$. The $-\nabla^2 \vec{E}$ was an important part of the wave equation. Since the curl of the curl formula is a general identity, we can write it with the vector potential.

$$\nabla \times (\nabla \times \vec{A}) = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$$

The convention is to choose $\nabla \cdot \vec{A} = 0$ for the vector potential. We have this freedom because the physics is in the curl: $\vec{B} = \nabla \times \vec{A}$.

$$\nabla \times \vec{B} = \nabla \times (\nabla \times \vec{A}) = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A} = 0 - \nabla^2 \vec{A}$$

$$\nabla \times \vec{B} = \mu_0 \vec{J} = -\nabla^2 \vec{A}$$

Compare with $\nabla^2 V = -\dfrac{\rho}{\varepsilon_0}$, which has solution $V(\vec{r}) = \dfrac{1}{4\pi\varepsilon_0} \displaystyle\int_\tau \frac{\rho(\vec{s})}{d} d\tau_S$

with $d = |\vec{r} - \vec{s}|$ and $d\tau_S \equiv d^3\vec{s}$ (to emphasize it's a scalar element).

Therefore, $\nabla^2 \vec{A} = -\mu_0 \vec{J}$ has solution $\vec{A}(\vec{r}) = \dfrac{\mu_0}{4\pi} \displaystyle\int_\tau \frac{\vec{J}(\vec{s})}{d} d\tau_S$.

V4. Inductance

1. Self-Inductance of Long Solenoid. We did this earlier. Neglect end effects.

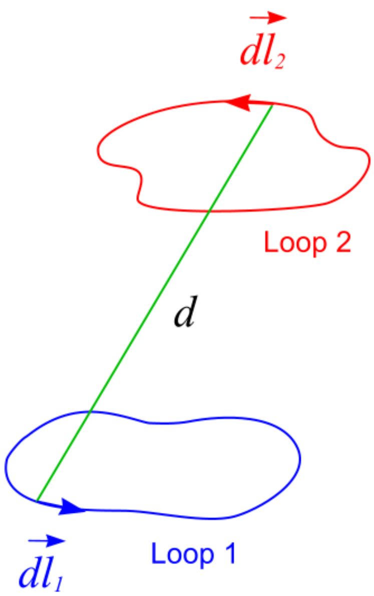
Summary: $B = \mu_0 nI$ $\quad L = \dfrac{\Phi_B}{I} = \dfrac{(nl)BA}{I} = \dfrac{(nl)(\mu_0 nI)A}{I} = \mu_0 n^2 Al$

Insert a material inside the coil. Replace μ_0 with μ.

$$\boxed{B = \mu nI} \qquad \boxed{L = \mu n^2 Al}$$

If we use $N = nl$, the total of number of windings, then $L = \dfrac{\mu N^2 A}{l}$.

2. Mutual Inductance. Here is why one loop affects another and vice versa.



For self-inductance: $\Phi = LI$.

For mutual inductance we work with the flux through Loop 2 due to the magnetic field produced by the current of Loop 1 when the switch is close.

$$\Phi_2 = M_{21} I_1$$

The notation M_{21} is the mutual inductance. We will show that $M_{12} = M_{21}$. The magnetic flux at loop 2 due to the magnetic field produced by loop 1 is

$$\Phi_2 = \iint \vec{B_1} \cdot \vec{da}_2 .$$

First introduce the vector potential: $\vec{B} = \nabla \times \vec{A}$.

$$\Phi_2 = \iint (\nabla \times \vec{A_1}) \cdot \vec{da}_2$$

Now use Stoke's Theorem.

$$\Phi_2 = \oint \vec{A_1} \cdot \vec{dl}_2$$

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int_{\tau} \frac{\vec{J}(\vec{s})}{d} d\tau_S$$

For current in a wire: $I = JA$.

$$\vec{J}(\vec{s})d\tau_S = J(A\overrightarrow{dl})$$

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{JA\overrightarrow{dl}}{d}$$

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{I\overrightarrow{dl}}{d}$$

For a current loop with current I :

$$\vec{A}(\vec{r}) = \frac{\mu_0 I}{4\pi}\oint \frac{\overrightarrow{dl}}{d}$$

Summary: $\Phi_2 = M_{21}I_1$ $\quad \Phi_2 = \oint \overrightarrow{A_1} \cdot \overrightarrow{dl}_2$ $\quad \vec{A}(\vec{r}) = \frac{\mu_0 I}{4\pi}\oint \frac{\overrightarrow{dl}}{d}$

$$\vec{A}_1(\vec{r}) = \frac{\mu_0 I_1}{4\pi}\oint \frac{\overrightarrow{dl_1}}{d} \qquad \Phi_2 = \oint \frac{\mu_0 I_1}{4\pi}\oint \frac{\overrightarrow{dl_1}}{d} \cdot \overrightarrow{dl}_2$$

$$M_{21} = \frac{\Phi_2}{I_1} = \oint \frac{\mu_0}{4\pi}\oint \frac{\overrightarrow{dl_1}}{d} \cdot \overrightarrow{dl}_2 = \frac{\mu_0}{4\pi}\oint\oint \frac{\overrightarrow{dl_1} \cdot \overrightarrow{dl}_2}{d}$$

$$\boxed{M_{21} = \frac{\mu_0}{4\pi}\oint\oint \frac{\overrightarrow{dl_1} \cdot \overrightarrow{dl}_2}{d}}$$ Neumann Formula

Note the symmetry in this formula. We can conclude that $M_{12} = M_{21}$.

3**. Mutual Inductance - Solenoid Inside a Solenoid.**

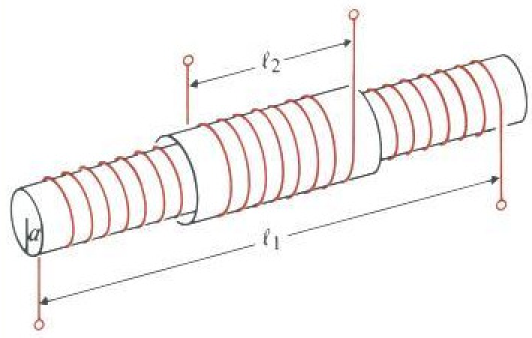


Figure Courtesy Department of Electrical & Computer Engineering, A. James Clark School of Engineering, University of Maryland

There are two coils with lengths l_1 and l_2, where $l_1 > l_2$. The number of turns for the inner coil, which is the longer one, is N_1 and the number of coils in the outer one is N_2.

The inner coil is wrapped around a cylindrical core with radius a, permeability μ, and has current I. We will use the formula for the infinitely long solenoid. The flux through a single loop of the outer coil due to the inner solenoid is found from the magnetic field of the inner solenoid

$$B = \mu n_1 I$$

and its area

$$A = \pi a^2 .$$

This gives

$$\Phi = BA = \mu n_1 I \pi a^2 .$$

Note that $n_1 = \dfrac{N_1}{l_1}$. Therefore, $\Phi = BA = \mu \dfrac{N_1}{l_1} I \pi a^2$.

But note that this is the flux through a single loop of the outer coil. Taking into consideration the N_2 turns, the flux linkage to the outer coil is

$$\Phi_2 = N_2 \Phi = \mu N_2 \frac{N_1}{l_1} I \pi a^2$$

To calculate the mutual inductance, we need to divide by the current.

$$M = \frac{\Phi_2}{I} = \mu \frac{N_1 N_2}{l_1} \pi a^2$$

V5. Optional Bonus Section (for those going to grad school)

Here is a derivation of a formula usually encountered in grad school. We already proved the following when we solved the Maxwell equations in free space to obtain the wave equation.

$$\nabla \times (\nabla \times \vec{A}) = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$$

Write this using Einstein's summation convention.

$$\nabla \times (\nabla \times \vec{A}) = \varepsilon_{ijk} \frac{\partial}{\partial x_i} (\nabla \times \vec{A})_j \hat{e}_k$$

$$(\nabla \times \vec{A})_j = \varepsilon_{nmj} \frac{\partial A_m}{\partial x_n}$$

$$\nabla \times (\nabla \times \vec{A}) = \varepsilon_{ijk} \frac{\partial}{\partial x_i} \varepsilon_{nmj} \frac{\partial A_m}{\partial x_n} \hat{e}_k = \varepsilon_{ijk} \varepsilon_{nmj} \frac{\partial^2 A_m}{\partial x_i \partial x_n} \hat{e}_k$$

Cycle the first j-index and swap partials to get $\nabla \times (\nabla \times \vec{A}) = \varepsilon_{kij} \varepsilon_{nmj} \frac{\partial^2 A_m}{\partial x_n \partial x_i} \hat{e}_k$.

This must give $\nabla \times (\nabla \times \vec{A}) = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$.

To get the first term from the Einstein version the following must fall out.

$$\varepsilon_{kij} \varepsilon_{nmj} \frac{\partial^2 A_m}{\partial x_n \partial x_i} \hat{e}_k => \frac{\partial^2 A_m}{\partial x_n \partial x_i} \hat{e}_k \delta_{kn} \delta_{im} = \frac{\partial}{\partial x_k} \frac{\partial A_m}{\partial x_m} \hat{e}_k = \nabla(\nabla \cdot \vec{A})$$

To get the second term from the Einstein version the following must also fall out.

$$\varepsilon_{kij} \varepsilon_{nmj} \frac{\partial^2 A_m}{\partial x_n \partial x_i} \hat{e}_k => -\frac{\partial^2 A_m}{\partial x_n \partial x_i} \hat{e}_k \delta_{km} \delta_{in} = -\frac{\partial^2 A_m}{\partial x_n \partial x_n} \hat{e}_m = -\nabla^2 \vec{A},$$

where there is a sum over then n-index since it appears twice.

$$\boxed{\varepsilon_{kij} \varepsilon_{nmj} = \delta_{kn} \delta_{im} - \delta_{km} \delta_{in}}$$

Can you come up with an easy way to remember this formula?