Theoretical Physics
Prof. Ruiz, UNC Asheville, doctorphys on YouTube
Chapter D Notes. "Derivation of the Maxwell Equations"

D0. Review of the Maxwell Equations

In the introductory physics course, second semester, one learns about the four Maxwell equations. If you are a non-physics major or took physics long ago, no problem. We are going to derive the equations anyway - well, sort of derive them. Here are the equations in two forms. Note that some physics texts give one integral sign for the area integrals.

$$\oiint \vec{E} \cdot \overrightarrow{dA} = \frac{Q}{\varepsilon_0}$$
$$\oiint \vec{B} \cdot \overrightarrow{dA} = 0$$
$$\oint \vec{B} \cdot \overrightarrow{dl} = \mu_0 i + \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt}$$
$$\oint \vec{E} \cdot \overrightarrow{dl} = -\frac{d\Phi_B}{dt}$$

$$\oiint \vec{E} \cdot \overrightarrow{dA} = \frac{Q}{\varepsilon_0}$$
$$\oiint \vec{B} \cdot \overrightarrow{dA} = 0$$
$$\oint \vec{B} \cdot \overrightarrow{dl} = \mu_0 i + \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt}$$
$$\oint \vec{E} \cdot \overrightarrow{dl} = -\frac{d\Phi_B}{dt}$$

We will also explain what each symbol means and talk about the math notation. So this class is really a review class. To connect to the previous chapter in our course, we will take a novel approach in arriving at the magnetic field B. Remember that our focus in this course is elegance and understanding at the fundamental level. Se we will not work the many applications you did in your intro course.

Classical Physics around 1800 included the following main laws of physics. These are Newton's Law of Motion in Classical Mechanics, Newton's Law of Universal Gravitation, and Coulomb's Law.

$$F = ma \qquad F_G = \frac{GMm}{r^2} \qquad F_E = \frac{kQq}{r^2}$$

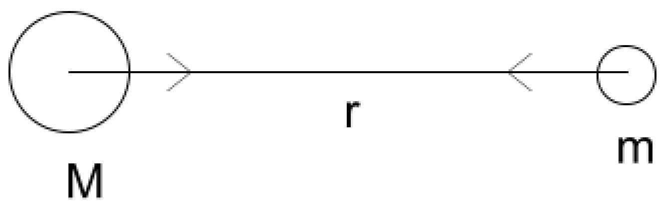
With gravity (our second equation), masses attract according to an inverse square law. Similarly with the electric force law (our third equation), the force between charges satisfies an inverse square law, but there you can have repulsion as well as attraction: "likes" (same sign for the charges) repel and "unlikes" (a plus charge and a minus charge) attract. We plan to derive the Maxwell equations from Coulomb's law and special relativity!

D1. Gauss's Law

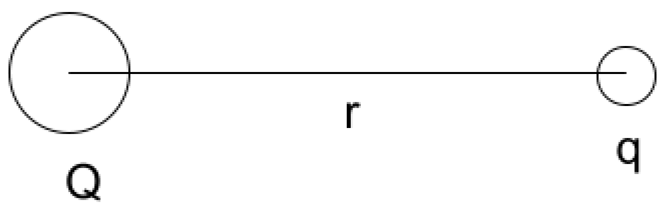
The first Maxwell equation is a restatement of Coulomb's Law in a form we call Gauss's Law. Coulomb's Law is

$$F_E = \frac{kQq}{r^2} \quad \text{where the constant} \quad k = \frac{1}{4\pi\varepsilon_0}$$

in the Meter-Kilogram-Second system of units. This form of the Metric System is often called the MKS system for short. The constant ε_0 is called the permittivity of free space. Note the similarity with the form of Newton's Law of Universal Gravitation below.



$$F_G = \frac{GMm}{r^2}$$



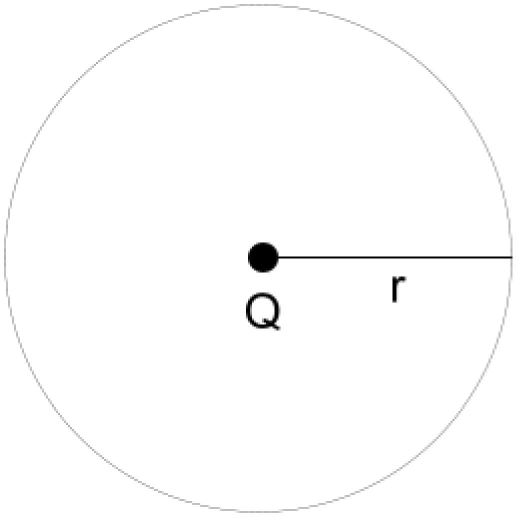
$$F_E = \frac{kQq}{r^2}$$

Both are inverse square laws. The Q represents charge, replacing the M which represents mass. The r is the distance between the centers of each mass or charge. Sometimes you see a minus sign in front of the gravity equation to remind us that the force is attractive. With the charges, when they are opposite in sign, you get the attraction.

Think about these properties called mass and charge. A fundamental force in nature means we endow matter with a property that goes with the force. For gravity it is mass M (or m). For the electric force, it is charge Q (or q). If you have mass, you experience gravity. If you have charge, you experience the electric force.

We define force fields for gravitation and the electric force by taking the smaller mass m to be 1 and the smaller charge q to be 1. Then we have forces per unit mass or charge:

$$g = \frac{GM}{r^2} \qquad\qquad E = \frac{kQ}{r^2}$$



Consider a charge Q at the origin and make a sphere at distance r to surround this charge.

The electric field at a distance r from the charge is

$$E = \frac{kQ}{r^2}$$

Since our force is a vector and will point outward for a positive test charge q, we write this in vector form as

$$\vec{E} = \frac{kQ}{r^2}\hat{r}$$

The $\hat{r}$ vector is a unit vector pointing away from the charge Q. Its precise direction depends on where you are on the sphere. This is unlike your unit vectors in Cartesian coordinates, $\hat{i}$, $\hat{j}$, and $\hat{k}$, which always point in the same directions.

We proceed to define a differential patch of area on the sphere and give it a unit vector direction outward. This is common practice with areas, i.e., to define area orientations with unit vectors perpendicular to the surfaces. When you come to think of this, that is the most conventional way to tell someone how to orient a plane piece of paper - by a unit vector perpendicular to the paper for the direction you want.

$$\overrightarrow{dA} = \hat{r}\, dA$$

Don't worry about the actual details for dA as we will not need explicit expressions. We will be talking at the most fundamental level for the most part.

We want to take $\vec{E} \cdot d\vec{A}$ and integrate over all the area on the sphere. When we integrate over a closed area we include a nice loop to emphasize that our area encloses on itself:

$$\oiint \vec{E} \cdot \overrightarrow{dA}$$

Let's do this integral. We have $\vec{E} = \frac{kQ}{r^2}\hat{r}$ and $\overrightarrow{dA} = \hat{r}\,dA$. Then,

$$\oiint \vec{E} \cdot \overrightarrow{dA} = \oint \frac{kQ}{r^2}\hat{r}\cdot\hat{r}\,dA .$$

The dot product $\hat{r}\cdot\hat{r}$ is equal to 1 since a dot product of any unit vector with itself is 1. The dot product between two vectors is the multiplication of the magnitudes times the cosine of the angle between them. The angle between a vector and itself is zero.

$$\oiint \vec{E} \cdot \overrightarrow{dA} = \oint \frac{kQ}{r^2}\,dA$$

Since we have a sphere here with a radius r that does not change and we want the surface area, we can pull out the constants:

$$\oiint \vec{E} \cdot \overrightarrow{dA} = \frac{kQ}{r^2}\oint dA .$$

This might still look scary but the integral is simply the surface area of a sphere. You know this. It is $A = 4\pi r^2$. So we get

$$\oiint \vec{E} \cdot \overrightarrow{dA} = \frac{kQ}{r^2}4\pi r^2 = 4\pi kQ .$$

Note that our constant $k = \frac{1}{4\pi\varepsilon_0}$. So we wind up with

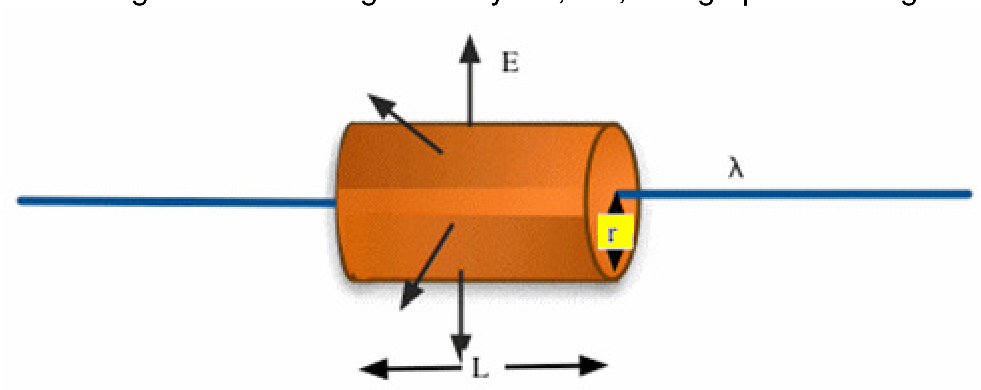
$$\oiint \vec{E} \cdot \overrightarrow{dA} = 4\pi kQ = 4\pi\frac{1}{4\pi\varepsilon_0}Q = \frac{Q}{\varepsilon_0}$$

This form is called Gauss's Law and it is our first Maxwell equation.

$$\oiint \vec{E} \cdot \overrightarrow{dA} = \frac{Q}{\varepsilon_0}$$

Note that Q is the charge inside the enclosure. This is an important point. We will see how to use this formula, including this idea below, which you saw in your intro physics course on electricity and magnetism (E&M). That would have been the second semester course of the one-year-long sequence.

Here is how you can apply this powerful formula to calculate the electric field due to an infinite line of charge that has charge density λ, i.e., charge per unit length.



Courtesy Department of Physics, Carleton College

You sketch an enclosed surface, a cylinder in this case, so that the electric field lines pierce the surface at 90°. Then for a distance r from the line, you apply Gauss's law. Since the E fields pierce the cylindrical part, we only worry about that part of the surface area. The dot product there will give 1 for the cosine while for the flat left and right sides the dot product will give zero.

$$\oiint \vec{E} \cdot d\vec{A} = \frac{Q}{\varepsilon_0}$$ becomes $$E(2\pi r L) = \frac{\lambda L}{\varepsilon_0}$$, which gives

$$E = \frac{1}{2\pi r}\frac{\lambda}{\varepsilon_0}$$

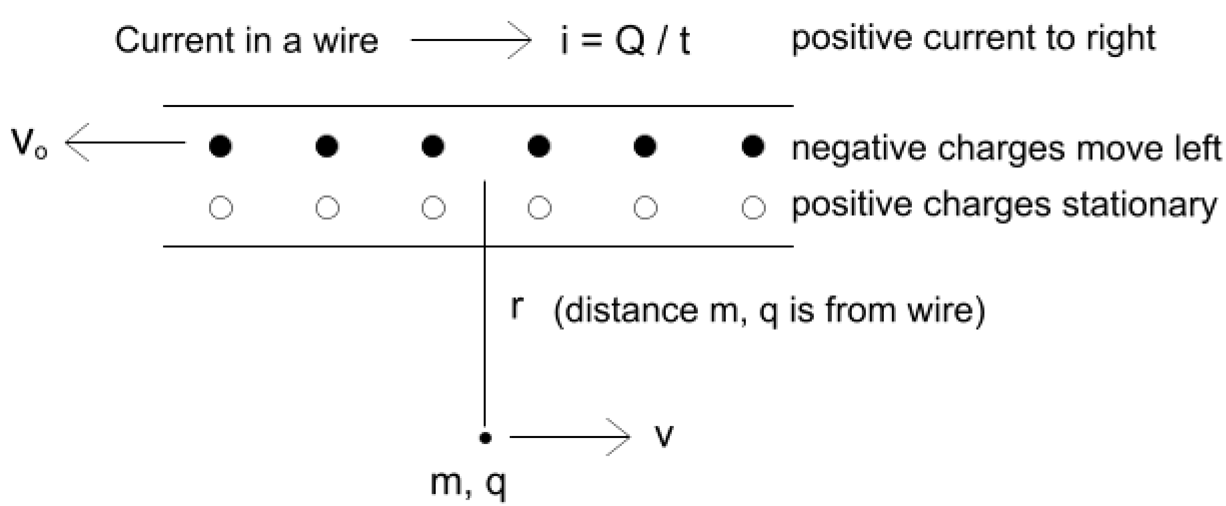
Note that the L dropped out. Pretty neat, right? A 1/r rule instead of inverse square since the line of charge is infinite and reinforces the field.

D2. The Magnetic Field

I encountered a form of what you are going to see here in Edward M. Purcell, *Electricity and Magnetism Berkeley Physics Course - Volume 2* (New York, McGraw-Hill, 1965).

We proceed to use the first Maxwell equation and relativity to derive the another Maxwell equation and part of the third one.

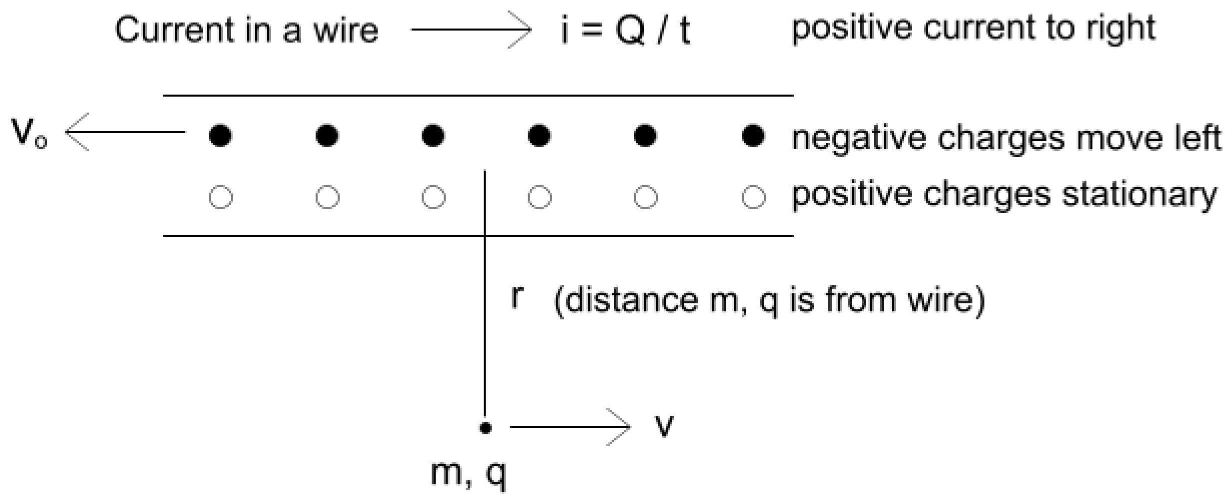
The black dots below are electrons moving to the left in a wire. We make things ideal and consider each electron moving to the left with equal spacing L_{lab}. The wire is neutral and our charge feels no force if at rest. But if the charge q moves, then the Lorentz contraction effect kicks in. The electrons are spaced closer to each other than the protons, as viewed from the moving frame, and the charge experiences a sideways force attracted to the electrons.



This is the idea. We did it. The rest is algebra. So for this chapter, your filling in steps are the practice problems and there is no homework to turn in for this chapter. We want you to have time reviewing your Maxwell equations and studying the notes well. In the above diagram, current is designated by i and the definition for current is charge per unit time. Current is moving charge. The convention is to take the flow in the positive direction, i.e., opposite to the direction of the moving negative charges.

Applying our line of charge formula $E = \dfrac{1}{2\pi r}\dfrac{\lambda}{\varepsilon_0}$ from the previous section, we have two lines of charge and write the magnitude of the electric field in the K' frame as

$E' = \dfrac{1}{2\pi r}\dfrac{\lambda_-' - \lambda_+'}{\varepsilon_0} > 0$ and the vector form is $\vec{E}' = E'\,\hat{j}$ (pointing up).



Here are our three frames:

Frame K is the laboratory frame.
Frame K' is the moving frame that goes along with the moving mass m (speed v).
Frame K" is the moving frame that rides along with the moving electrons (sped v_o).

Moving mass m "sees" positive charge (the white circles) moving to the left at $\beta = \frac{v}{c}$

Moving mass m "sees" negative charge (the black dots) moving left at $\beta_T = \frac{v_T}{c}$

The speed β_T is found from the relativistic addition of speeds v and v_0. This result is

$$\beta_T = \frac{\beta_0 + \beta}{1 + \beta_0 \beta}.$$

For our $E' = \frac{1}{2\pi r} \frac{\lambda_-' - \lambda_+'}{\varepsilon_0} > 0$, we apply the generic formula for linear charge density $\lambda = \frac{Q}{L}$. i.e., charge per length. In the lab frame we take $\frac{Q}{L_{lab}}$, where each charge is Q and the spacing is L_{lab}. In the lab frame negative and positive cancel and the wire is neutral as the electrons move to the left in step a distance L_{lab} apart. In the K" frame we take the electron separation distance to be L''.

Therefore for $E' = \frac{1}{2\pi r}\frac{\lambda_-' - \lambda_+'}{\varepsilon_0} > 0$ we need λ_-' and λ_+'. From the K' the Lorentz contraction for the positive charges is $L_+' = L_{lab}\sqrt{1-\beta^2}$. For the electrons we use the electron frame length times its contraction: $L_-' = L''\sqrt{1-\beta_T{}^2}$. Then

$$\lambda_-' = \frac{Q}{L''\sqrt{1-\beta_T{}^2}} \text{ and } \lambda_+' = \frac{Q}{L_{lab}\sqrt{1-\beta^2}}$$

For the electric field as seen in the K' frame, i.e., the moving mass, we have

$$E' = \frac{1}{2\pi r\varepsilon_0}\left[\frac{Q}{L''\sqrt{1-\beta_T{}^2}} - \frac{Q}{L_{lab}\sqrt{1-\beta^2}}\right]$$

But we want the laboratory spacing here. We want to get rid of that L'', which is the electron spacing in its own K" frame. From the laboratory perspective, the electron spacing is seen to be contracted as

$$L_{lab} = L''\sqrt{1-\beta_0{}^2}\ .$$

We substitute $\frac{1}{L''} = \frac{\sqrt{1-\beta_0{}^2}}{L_{lab}}$ into our E' equation and obtain

$$E' = \frac{1}{2\pi r\varepsilon_0}\frac{Q}{L_{lab}}\left[\frac{\sqrt{1-\beta_0{}^2}}{\sqrt{1-\beta_T{}^2}} - \frac{1}{\sqrt{1-\beta^2}}\right]$$

This looks complicated. But we will get a Maxwell equation and a half out of this. We will rewrite this equation at the top of the next page and continue.

$$E' = \frac{1}{2\pi r\varepsilon_0}\frac{Q}{L_{lab}}\left[\frac{\sqrt{1-\beta_0{}^2}}{\sqrt{1-\beta_T{}^2}} - \frac{1}{\sqrt{1-\beta^2}}\right]$$

Using the relativistic addition of velocities β_0 and β for β_T,

$$E' = \frac{1}{2\pi r\varepsilon_0}\frac{Q}{L_{lab}}\left[\frac{\sqrt{1-\beta_0{}^2}}{\sqrt{1-(\frac{\beta_0+\beta}{1+\beta_0\beta})^2}} - \frac{1}{\sqrt{1-\beta^2}}\right]$$

Focus on part of the math here.

$$\sqrt{1-(\frac{\beta_0+\beta}{1+\beta_0\beta})^2} = \sqrt{\frac{1+2\beta_0\beta+\beta_0{}^2\beta^2-\beta_0{}^2-2\beta_0\beta-\beta^2}{(1+\beta_0\beta)^2}}$$

$$\sqrt{1-(\frac{\beta_0+\beta}{1+\beta_0\beta})^2} = \sqrt{\frac{1+\beta_0{}^2\beta^2-\beta_0{}^2-\beta^2}{(1+\beta_0\beta)^2}} = \sqrt{\frac{(1-\beta_0{}^2)(1-\beta^2)}{(1+\beta_0\beta)^2}}$$

$$E' = \frac{1}{2\pi r\varepsilon_0}\frac{Q}{L_{lab}}\left[\frac{(1+\beta_0\beta)\sqrt{1-\beta_0{}^2}}{\sqrt{(1-\beta_0{}^2)(1-\beta^2)}} - \frac{1}{\sqrt{1-\beta^2}}\right]$$

$$E' = \frac{1}{2\pi r\varepsilon_0}\frac{Q}{L_{lab}}\left[\frac{(1+\beta_0\beta)}{\sqrt{1-\beta^2}} - \frac{1}{\sqrt{1-\beta^2}}\right]$$

$$E' = \frac{1}{2\pi r\varepsilon_0}\frac{Q}{L_{lab}\sqrt{1-\beta^2}}\left[(1+\beta_0\beta)-1\right]$$

$$E' = \frac{1}{2\pi r\varepsilon_0}\frac{Q}{L_{lab}\sqrt{1-\beta^2}}\beta_0\beta$$

The mess cleared itself up big time!

$$E' = \frac{1}{2\pi r\varepsilon_0} \frac{Q}{L_{lab}\sqrt{1-\beta^2}} \beta_0 \beta$$

The force is

$$F_y' = qE' = \frac{q}{2\pi r\varepsilon_0} \frac{Q}{L_{lab}\sqrt{1-\beta^2}} \beta_0 \beta \; .$$

Note that this is a y force. But all our frames of reference move horizontally. Therefore, the y-momentum is the same in general. Therefore we can write in general

$$F_y = \frac{dp_y}{dt} \quad \text{and} \quad F_y' = \frac{dp_y}{dt'}$$

It is the time that is different: $dt = \frac{dt'}{\sqrt{1-\beta^2}}$, the time dilation we saw earlier. Since $dt' = \sqrt{1-\beta^2}\, dt$, we have $F_y' = \frac{dp_y}{dt'} = \frac{1}{\sqrt{1-\beta^2}} \frac{dp_y}{dt} = \frac{F_y}{\sqrt{1-\beta^2}}$. So in the lab frame we have

$$F_y = qE = \frac{q}{2\pi r\varepsilon_0} \frac{Q}{L_{lab}} \beta_0 \beta$$

$$F_y = qE = \frac{q}{2\pi r\varepsilon_0} \frac{Q}{L_{lab}} \frac{v_0}{c} \frac{v}{c}$$

$$F_y = \frac{q}{2\pi r} \left[\frac{1}{\varepsilon_0 c^2} \right] \left[\frac{Q}{L_{lab}} v_0 \right] v$$

Now, this grouping $\left[\frac{1}{\varepsilon_0 c^2} \right]$ is defined to be $\frac{1}{\varepsilon_0 c^2} \equiv \mu_0$

$$F_y = \frac{q}{2\pi r}\mu_0\left[\frac{Q}{L_{lab}}v_0\right]v\,.$$

This constant μ_0 is the magnetic permeability or simply the permeability of the vacuum or free space. It is analogous the ε_0, the permittivity of the vacuum or free space.

The next grouping $\frac{Q}{L_{lab}}v_0$ is current since $v_0 = \frac{L_{lab}}{t}$ and $\frac{Q}{L_{lab}}v_0 = \frac{Q}{L_{lab}}\frac{L_{lab}}{t} = \frac{Q}{t}$. Note that earlier in our analysis we called the current by the letter i. This can be confusing since i also stands for the square root of minus one. This bothers electrical engineers so much that they define the square root of minus one to be j because current must be i for them.

So now we have

$$F_y = \frac{q}{2\pi r}\mu_0\left[\frac{Q}{L_{lab}}v_0\right]v = \frac{q}{2\pi r}\mu_0\left[\frac{Q}{L_{lab}}\frac{L_{lab}}{t}\right]v = \frac{q}{2\pi r}\mu_0\left[\frac{Q}{t}\right]v = \frac{q}{2\pi r}\mu_0\, i\, v$$

Write this as

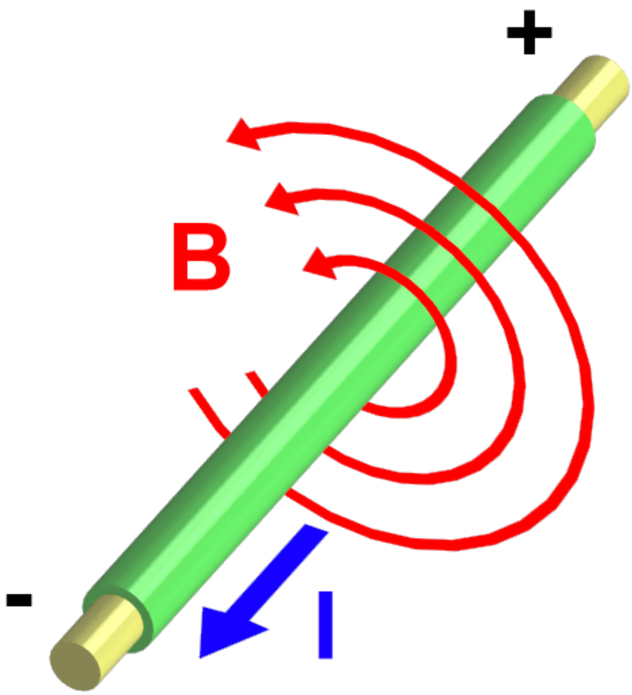
$$F_y = qv\frac{\mu_0 i}{2\pi r}\,.$$

We have isolated the two parameters on the left that deal with properties of the moving charge. The rest of the pieces relate to the external force field. We call this a magnetic field and designate it with the letter B

$$B = \frac{\mu_0 i}{2\pi r}\,.$$

The magnetic field is due to the current in the wire. Due to the cylindrical symmetry we can assign the unit vector $\hat{\theta}$. To get a sense of this direction, use your right hand with thumb aligned with the current. The B field then takes on the direction of your curved fingers.

$$\vec{B} = \frac{\mu_0 i}{2\pi r}\hat{\theta}\,.$$



Here is a figure (Courtesy Wikimedia) showing the right-hand rule in action to get the direction of the magnetic field.

$$\vec{B} = \frac{\mu_0 i}{2\pi r}\hat{\theta}$$

The magnitude of our force is

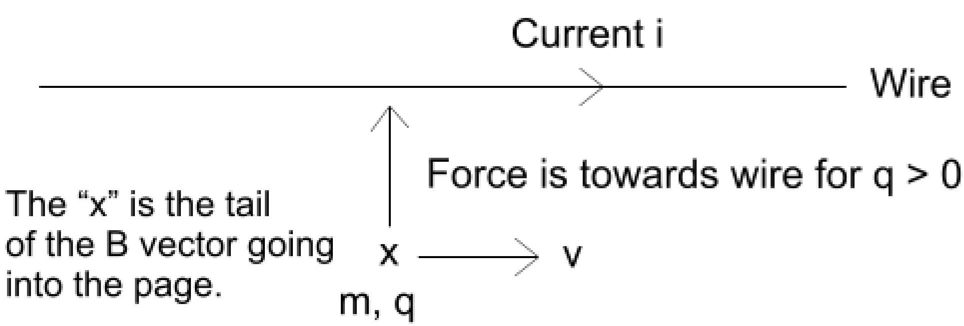
$$F_y = qv\frac{\mu_0 i}{2\pi r} = qvB .$$

Since the force is upward towards the wire we have

$\vec{F} = qvB\,\hat{j}$. With $\vec{B} = \dfrac{\mu_0 i}{2\pi r}\hat{\theta}$ and $\vec{v} = v\,\hat{i}$ along the x direction, we can use a cross product to express the force:

$$\vec{F} = q\vec{v}\times\vec{B}\text{, where } \vec{B} = \frac{\mu_0 i}{2\pi r}\hat{\theta} .$$

We designate the B vector below with an "x" to represent the tail of the vector pointing into the page. The cross product gets you the force in the right direction.



The general force law which includes both electric and magnetic fields is called the Lorentz force law, named after Lorentz of Lorentz transformation fame.

$$\vec{F} = q\vec{E} + q\vec{v}\times\vec{B}$$

We can express the magnetic field B as a loop line integral wrapping around the wire. Let's do this backwards. We then arrive at Ampère's Law in integral form.

$$B = \frac{\mu_0 i}{2\pi r}, \qquad B(2\pi r) = \mu_0 i, \quad \text{and} \quad \oint \vec{B} \cdot \overrightarrow{dl} = \mu_0 i .$$

This is the integral we can use to calculate B for the current in the wire. You did this in your introductory physics class. Since the magnetic field lines wrap around on themselves, if we swallow up a magnet like we did for a charge in Gauss's Law, we get zero. There are no piercings of the magnetic field outward through the surface.

$$\oiint \vec{B} \cdot \overrightarrow{dA} = 0$$

This is the second Maxwell equation. Here is a summary of what we have so far where we put the surface integrals first. The first is Gauss's Law, an alternate form of Coulomb's Law. The third is Ampère's Law. The second law is not associated with anyone in particular. But it involves the magnetic field and Ampere is a key figure.

$$\boxed{\oiint \vec{E} \cdot \overrightarrow{dA} = \frac{Q}{\varepsilon_0} \qquad \oiint \vec{B} \cdot \overrightarrow{dA} = 0 \qquad \oint \vec{B} \cdot \overrightarrow{dl} = \mu_0 i}$$

$$\boxed{\vec{F} = q\vec{E} + q\vec{v} \times \vec{B}}$$

Charles Augustin de Coulomb (1736-1806)	Johann Carl Friedrich Gauss (1777-1855)	André Marie Ampère (1775-1836)
		

Courtesy School of Mathematics and Statistics, University of St. Andrews, Scotland

D3. Faraday's Law

Michael Faraday (1791-1867)
Courtesy School of Mathematics and Statistics
University of St. Andrews, Scotland

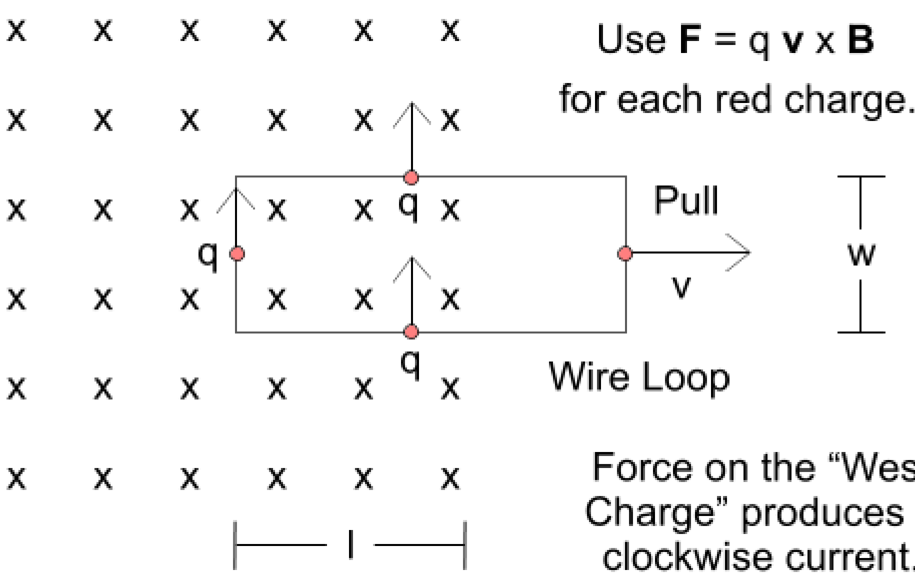
Here is Faraday's Law, which you encountered in intro physics.

$$\oint \vec{E} \cdot \vec{dl} = -\frac{d\Phi_B}{dt},$$

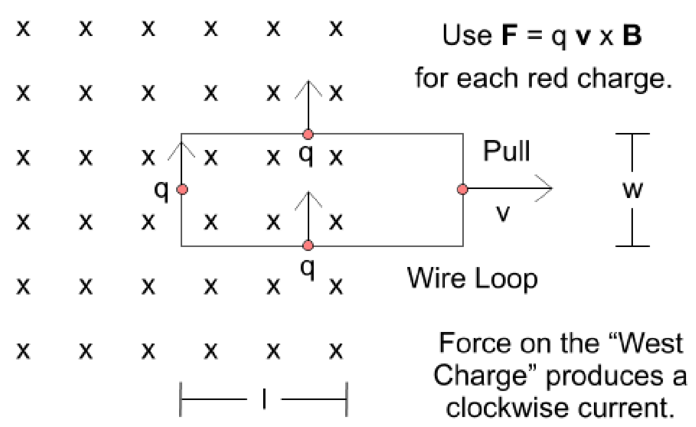
where Φ_B is the magnetic flux. Magnetic flux is found by multiplying the magnetic field with the area through which the field lines penetrate.

$$\Phi_B = BA$$

If the magnetic field is not constant, you have to do an integral. Let's see if we can understand a theoretical argument as to why this is true. We start from what we know. The "x" marks below are the tails of the constant magnetic field B lines that goes into the page. We pull a wire loop through this field.



Apply $\vec{F} = q\vec{v} \times \vec{B}$ to each of the four red positive charges in the wire. These is no force on the east charge since B = 0 there. The other charges are pulled upward but only the west charge starts to move to produce a current due to $F = qvB$. An electric field is generated since we have induced current. The electric field generated must be $E = vB$ since $F = qvB = qE$.



The velocity seen in our formula

$$E = vB$$

is given by $v = -\frac{dl}{dt}$, which is the negative of the decrease in our length portion where there is the magnetic field.

This gives

$$E = vB = -\frac{dl}{dt}B .$$

Now consider the loop integral for this generated E field. The only relevant side is the west side.

$$\oint \vec{E} \cdot \vec{dl} = Ew .$$

On the west side the electric field lines up with the differential vector length element. We integrate along the path where the electric field is pointing. So we integrate up and therefore get the positive Ew.

The integral for the top part is zero since the E field is perpendicular to the direction which at the top is to the right. There is no E field on the east side. The integral at the bottom is zero similar to the top analysis.

Putting it all together, we obtain

$$\oint \vec{E} \cdot \vec{dl} = -\frac{dl}{dt}Bw .$$

To allow for pulling the wire upwards, we move w into the derivative. The w is constant here but would not be if we pulled upwards instead of the right.

$$\oint \vec{E} \cdot \vec{dl} = -\frac{d(lw)}{dt} B \ .$$

Note that the product lw is the length times width for where the magnetic field penetrates through the wire. So we call this the area A = lw and write

$$\oint \vec{E} \cdot \vec{dl} = -\frac{d(lw)}{dt} B = -\frac{dA}{dt} B \ .$$

Since the B is constant we can pull the B into the derivative. But this is significant because if we increased the B field instead of moving the wire, we would get the same effect.

$$\oint \vec{E} \cdot \vec{dl} = -\frac{d(BA)}{dt} \ .$$

Since $\Phi_B = BA$, the magnetic flux, we have arrived at Faraday's Law from a theoretical analysis.

$$\oint \vec{E} \cdot \vec{dl} = -\frac{d\Phi_B}{dt}$$

Now we have four equations, but we have to add one important piece. We do that in the final section.

$$\oiint \vec{E} \cdot \vec{dA} = \frac{Q}{\varepsilon_0}$$

$$\oiint \vec{B} \cdot \vec{dA} = 0$$

$$\oint \vec{B} \cdot \vec{dl} = \mu_0 i$$

$$\oint \vec{E} \cdot \vec{dl} = -\frac{d\Phi_B}{dt}$$

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$

D4. The Displacement Current

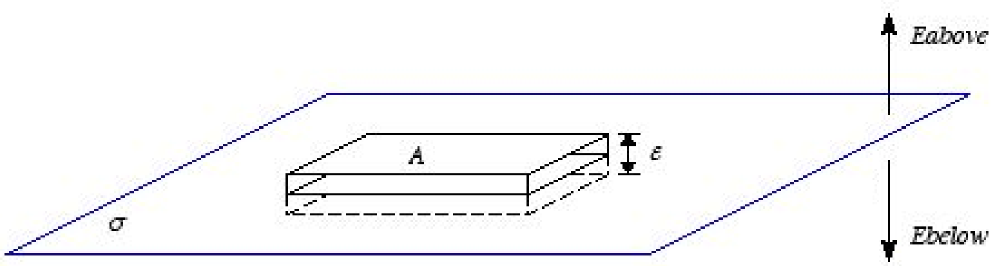
James Clerk Maxwell (1831-1879)
Courtesy School of Mathematics and Statistics
University of St. Andrews, Scotland

Maxwell focused on the fact that a changing magnetic flux produces an electric field. For cases when the magnetic field strength increases or decreases through a given area one can say that a changing magnetic field produces an electric field.

Could the reverse be true? Could a changing electric field produce a magnetic field? Could a changing electric flux mean we get a magnetic field?

He found that to be the case, which made possible electromagnetic waves, which we study later.

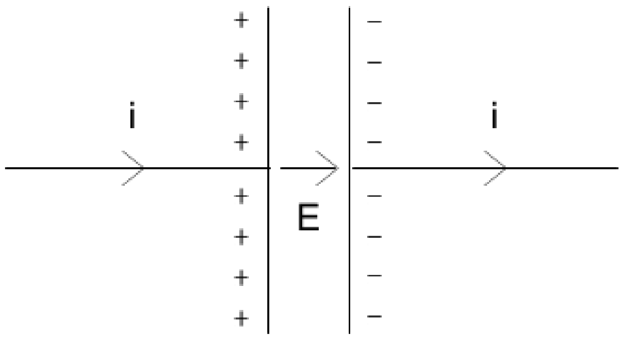
Before we analyze Maxwell's hunch, we will need a calculation you did in your intro physics course with Gauss's Law. To be complete, we will repeat it here. The problem is to find the electric field due to an infinite sheet of charge with density σ per unit area. We make a little rectangle box to enclose a piece of the plane inside.



Courtesy Prof. Frank L. H. Wolfs, Department of Physics
and Astronomy, University of Rochester, NY

We apply Gauss's Law $\oiint \vec{E} \cdot \overrightarrow{dA} = \dfrac{Q}{\varepsilon_0}$, where Q is the charge inside. The electric field is upward on the above surface and downward on the below surface. The result is

$$EA + EA = \frac{\sigma A}{\varepsilon_0} \text{, which gives } E = \frac{\sigma}{2\varepsilon_0}.$$



Earlier we worked out the magnetic field a distance r from a wire with current i using Ampère's law

$$\oint \vec{B} \cdot \vec{dl} = \mu_0 i$$

and found $B = \dfrac{\mu_0 i}{2\pi r}$.

What happens if we interrupt the current by placing two plates each with area A in the path? There is no current across the gap. Do we get B = 0 outside these plates? The two parallel plates make up a capacitor, a circuit element that can store charge. Those plates are getting charged up and the plates stop the current from going across the gap in the middle.

The electric field inside is the sum of two sheets of charge. We neglect the edge effects. Since each sheet produces $E = \dfrac{\sigma}{2\varepsilon_0}$ and the opposite charges on each side work together to produce an even stronger electric field, the total strength due to both sheets is $E = \dfrac{\sigma}{\varepsilon_0}$, i.e., double. The charge density is $\sigma = \dfrac{Q}{A}$, where Q is the absolute magnitude of the total charge on each plate and the area of each plate is A.

In the spirit of Maxwell's insight, we calculate the change in electric flux between the plates.

$$\frac{d\Phi_E}{dt} = \frac{d(EA)}{dt} = \frac{d}{dt}\left[\frac{\sigma}{\varepsilon_0} A\right] = \frac{d}{dt}\left[\frac{Q}{\varepsilon_0}\right] = \frac{i}{\varepsilon_0}.$$

We make the bold statement that the magnetic field B should be the same outside the plates too. Then we need $\oint \vec{B} \cdot \vec{dl} = \mu_0 i$ here too. But there is no actual i. Instead we have $\dfrac{d\Phi_E}{dt} = \dfrac{i}{\varepsilon_0}$. So we need $\mu_0 \varepsilon_0 \dfrac{d\Phi_E}{dt} = \mu_0 \varepsilon_0 \dfrac{i}{\varepsilon_0} = \mu_0 i$ to make it work.

The result is to add this piece to Ampère's Law:

$$\oint \vec{B} \cdot \vec{dl} = \mu_0 i + \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt} .$$

The added term is sometimes called the displacement current but that is not a good name for it. The four equations are now complete and named after Maxwell for adding this crucial term.

The Maxwell Equations and the Lorentz Force Law

$$\oiint \vec{E} \cdot \vec{dA} = \frac{Q}{\varepsilon_0}$$

$$\oiint \vec{B} \cdot \vec{dA} = 0$$

$$\oint \vec{B} \cdot \vec{dl} = \mu_0 i + \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt}$$

$$\oint \vec{E} \cdot \vec{dl} = -\frac{d\Phi_B}{dt}$$

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$