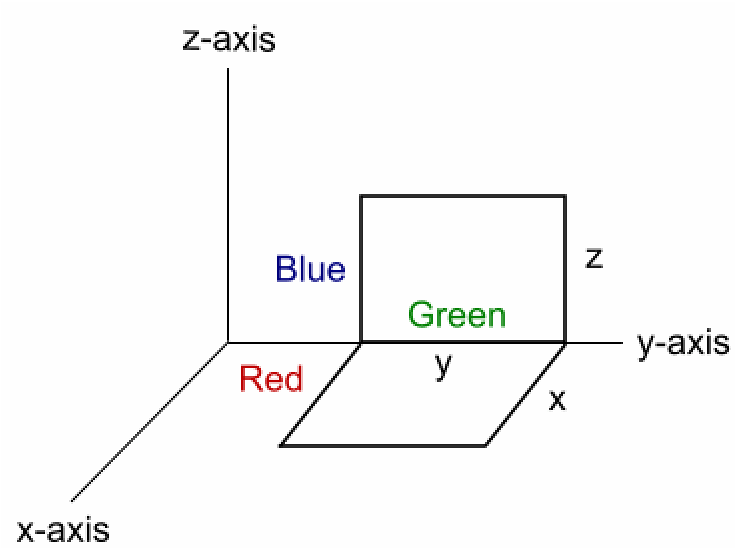
**Theoretical Physics**
**Prof. Ruiz, UNC Asheville**
**Chapter I Homework. Quantum Mechanics**

**HW-I1. Undetermined Multipliers.** Use Lagrange's Method of Undetermined Multipliers to solve this problem. A group of science and math students have a small area in an assembly room where they get to write on the wall and floor.




The wall area has dimensions y by z and the floor area has dimensions x by y.

To mark off these regions, you need to use three strips of colors as follows: a red strip of narrow tape parallel to the x-axis, a green strip of narrow tape along the y-axis, and a blue strip of narrow tape parallel to the z-axis.

Note that there is one strip of each color. The strips do NOT enclose the areas. They just mark off lengths in each of the three dimensions. However, these lengths define your two areas.

The students want to maximize the total area they can define with their strips.

However, the group must pay for this space as follows:

a) The blue strip is 2 cents per centimeter (cm).
b) The green strip is 1 cent per cm.
c) The red strip is free.

And there are two catches:

a) The sum length of all your strips must be 100 cm,
b) You have 100 cents to spend and you must spend all your money.

Find x, y, and z using the Method of Undetermined Multipliers.

**Proceed to the Next Page for HW-I2.**

**HW-I2. Two Quantum States.**

a) The Sketch. Sketch a dotted horizontal line to represent a zero reference energy point. Then draw a short solid horizontal line below this reference and label this line with energy $\varepsilon_1 = -\varepsilon$. Then draw a short solid horizontal line above your reference and label this line with energy $\varepsilon_2 = +\varepsilon$.

b) The Partition Function. At temperature $T$ give the partition function for this system.

c) Occupation Numbers. Then give the most probable occupation numbers $n_1$ and $n_2$ for each of these states if the total number of particles is some large value $N = n_1 + n_2$.

d) Energy. Give the average energy $\overline{E} = \dfrac{n_1\varepsilon_1 + n_2\varepsilon_2}{n_1 + n_2}$ as a hyperbolic function and give a sketch of the average energy as a function of temperature.

e) What are the occupation numbers and average energy in the limit as the temperature approaches absolute zero, i.e., $T \to 0$.

f) What are the occupation numbers and average energy in the limit as the temperature approaches infinity, i.e., $T \to \infty$.

g) Compare your results for (e) and (f) in terms of entropy, i.e., the disorder of the system, using words. Why do you expect your answers for (e) and (f) to come out the way they do in terms of entropy?

The point values for this problem are not equally distributed among the parts. Most weight is assigned to the energy formula, sketch, and entropy discussion of the two limiting cases.

**Proceed to the Next Page for HW-I3.**

**HW-I3. Infinite Quantum States.** Consider Max Planck's discrete energy levels given by $E_n = nhf$ . Give the partition function as a summation from zero to infinity for these states.

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This section need not be included in your homework write-up.

To evaluate your petition function, note that your sum is of the form

$$Z = 1 + e^{-x} + e^{-2x} + e^{-3x} + \ldots \text{, where } x > 0 .$$

Let $r = e^{-x}$ . Then

$$Z = 1 + r + r^2 + r^3 + \ldots \quad \text{and} \quad rZ = r + r^2 + r^3 + \ldots .$$

Each series has an infinite number of terms, each getting smaller and smaller since $r = e^{-x} < 1$ . Subtracting your two infinite sums,

$$Z - rZ = 1 , \quad Z(1-r) = 1 , \quad \text{and} \quad Z = \frac{1}{1-r} .$$

If you have never seen this, it is highly recommend that you derive the result for a finite sum of n terms: $S_n = a + ar + ar^2 + \ldots + ar^{n-1} = a\left[\frac{1-r^n}{1-r}\right]$.

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Use the following derivative trick in order to calculate your average energy.

$$\sum_n ne^{-nx} = -\frac{d}{dx}\sum_n e^{-nx} = -\frac{d}{dx}\left[\frac{1}{1-e^{-x}}\right]$$

Your final answer will be the prize: $\overline{E} = \dfrac{hf}{e^{\frac{hf}{kT}} - 1}$ .

You get the classical result by making "h" small, just as you get the classical formulas from relativity by making "c" large. What is the limit for the energy when h goes to zero if you do a Taylor's expansion using $e^z \approx 1 + z$ ? The equipartition theorem in classical thermodynamics says you get $kT/2$ for each degree of freedom, i.e., each way the system can absorb energy. Remember the ideal gas with 3kT/2? Here, an oscillator has one degree for kinetic energy along the direction of the "spring" and one degree for its potential energy. So you should get kT for your classical average energy.