Theoretical Physics
Prof. Ruiz, UNC Asheville
Chapter K Homework. The Pauli Equation

HW-K1. Matrix in an Exponential. We define a matrix in an exponential as follows:

$$e^A = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \ldots$$, where $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and

$A^2 = AA$, $A^3 = AAA$, and so on (matrix multiplication).

Find the simplest single 2 x 2 matrix result for $e^{i\theta\sigma_x}$ where $\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Hint: You are obviously not expected to calculate an infinite amount of matrix multiplications as you would never finish your program at UNCA. Instead, you will see patterns. To appreciate these patterns, first play around with various multiplications of $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ with itself over and over again.

HW-K2. Wave Function with Spin. First note the definition for multiplying a row matrix with a column matrix.

$$\begin{bmatrix} a & b \end{bmatrix}\begin{bmatrix} c \\ d \end{bmatrix} = ac + bd$$. You will need this definition.

Dirac defined a "bra" and "ket" from breaking apart the word "bracket." A bracket can be visualized as $<>$.

The "ket" is $|\psi\rangle \equiv \begin{bmatrix} f(x) \\ g(x) \end{bmatrix}$ and the "bra" is $\langle\psi| \equiv \begin{bmatrix} f^*(x) & g^*(x) \end{bmatrix}$.

For a matrix $A = \begin{bmatrix} a_{11}(x) & a_{12}(x) \\ a_{21}(x) & a_{22}(x) \end{bmatrix}$,

$$\langle \psi | A | \psi \rangle \equiv \int_{-\infty}^{\infty} [f^* \quad g^*] \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} f \\ g \end{bmatrix} dx$$

One works out the matrix multiplications and then integrates over all x.

Consider the wave function ψ :

$$\psi = 0 \text{ for } -\infty \le x < 0$$

$$\psi = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-x/2} \text{ for } 0 \le x \le \infty .$$

and the Hermitian matrix operator:

$$A = \begin{bmatrix} x & i \\ -i & x \end{bmatrix}.$$

Using the **derivative trick** at some point, calculate

$$\langle \psi | A | \psi \rangle .$$

Special Grading Rubric for this problem:

3 points for applying the appropriate derivative trick correctly;
3 points for the correct matrix multiplication;
2 points for the correct final numerical answer;
2 points for proper notation at all places.