Theoretical Physics
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Chapter L Homework. The Dirac Equation

HW-L1 and HW-L2. The SU2) Generators (NOTE - WORTH TWO HOMEWORK PROBLEMS). Using the **Guide** below with this question, show that the general form for the SU(2) matrices can be expressed in the form

$$U = e^{i\alpha \hat{n}\cdot\vec{\sigma}}.$$

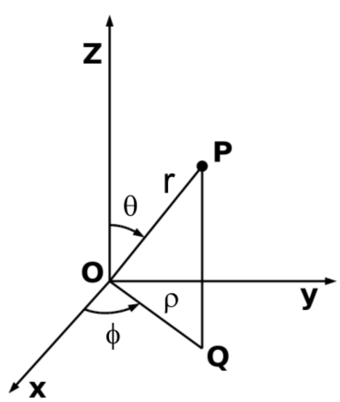
The Paul matrices are said to be the generators for the SU(2) group.

Guide. You can start from the general form for a x 2 x unitary matrix in Chapter J.

$$U = a_r \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + ib_i \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + ib_r \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} + +ia_i \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

where $\det(U)$ forces $a_r^{\,2} + a_i^{\,2} + b_r^{\,2} + b_i^{\,2} = 1$.

There are 4 parameters, but when we apply the condition that the determinant is 1 we reduce this to three free parameters. We will be using spherical coordinates (r,θ,ϕ) defined in the figure below;



$$x = \rho\cos\phi \quad y = \rho\sin\phi \quad \rho = r\sin\theta$$

$$x = r\sin\theta\cos\phi$$
$$y = r\sin\theta\sin\phi$$
$$z = r\cos\theta$$

$$\vec{r} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k} = r\sin\theta\cos\phi\,\hat{i} + r\sin\theta\sin\phi\,\hat{j} + r\cos\theta\,\hat{k}$$

$$\hat{n} \equiv \hat{r} = \sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k}$$, taking $r = 1$.

$$\hat{n}\cdot\vec{\sigma} = \sin\theta\cos\phi\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} + \sin\theta\sin\phi\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix} + \cos\theta\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$$

Introduce for the above b_i , b_r , and a_i new parameters b , θ , and ϕ so that

$$b_i = b\sin\theta\cos\phi,\ b_r = b\sin\theta\sin\phi, \text{ and } a_i = b\cos\theta.$$

Use these substitutions to arrive at

$$U = a_r\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + ib\,\hat{n}\cdot\vec{\sigma} = a_r I + ib\,\hat{n}\cdot\vec{\sigma}.$$

Show that the solution $a_r = \cos\alpha$ and $b = \sin\alpha$ insures that $\det(U) = 1$.

Your three free parameters are now α , θ , and ϕ .

Show that $(\hat{n}\cdot\vec{\sigma})^{2k} = I$ and $(\hat{n}\cdot\vec{\sigma})^{2k+1} = \hat{n}\cdot\vec{\sigma}$ for $k = 0, 1, 2, \ldots$

Use these results with $U = a_r I + ib\,\hat{n}\cdot\vec{\sigma}$ to show that $U = e^{i\alpha\,\hat{n}\cdot\vec{\sigma}}$.

Summary Comments of Your Achievement

You have found the exponential map between a Lie Algebra (described by the commutators of the Pauli matrices) and a Lie Group (all the matrices in the special unitary group). The Pauli matrices alone can generate all the SU(2) matrices thanks to the $\det(U) = 1$ condition, which essentially eliminates one of the original four free parameters of the unitary condition. The Pauli matrices are generators of the group SU(2). They generate all the SU(2) matrices by the following exponential relationship.

$$U = e^{i\alpha\,\hat{n}\cdot\vec{\sigma}}$$

The three free parameters are α , θ , and ϕ . The generators satisfy a Lie Algebra:

$$\left[\sigma_j, \sigma_k\right] = 2i\varepsilon_{jkl}\,\sigma_l.$$

Now you have deep insight into the relationship between a Lie Algebra and its associated group.

HW-L3. Show that $\alpha_2\alpha_3 + \alpha_3\alpha_2 = 0$ by explicitly multiplying out the 4 x 4 matrices. Then use the shortcut method with $\alpha_2 = \begin{bmatrix} 0 & \sigma_y \\ \sigma_y & 0 \end{bmatrix}$ and $\alpha_3 = \begin{bmatrix} 0 & \sigma_z \\ \sigma_z & 0 \end{bmatrix}$ to show the same result. The zeros in these matrices are understood to be $0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. When you work this problem out you are multiplying 4 x 4 matrices by considering four 2 x 2 subsections of each matrix. See the last page of the text, Chapter L, to see an example worked out using the shortcut method.