Theoretical Physics
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Chapter R Homework. Convolution

R1. Convolution. You will work with the following two functions.

$$f(t) = t \quad \text{and} \quad g(t) = t^2$$

a. Show that the convolution for these functions

$$f(t) * g(t) = \int_0^t f(u)g(t-u)du = \frac{t^4}{12}$$

b. Show that if you calculate the convolution by the following formula, you get the same result. Since this is true in general, this means that the convolution operation is commutative.

$$g(t) * f(t) = \int_0^t g(u)f(t-u)du = \frac{t^4}{12}$$

R2. Laplace Transform of a Convolution. Show for the above case that the Laplace transform of the convolution is given by the product of the Laplace transforms of each function.

$$L\{g(t) * f(t)\} = L\{g(t)\}L\{f(t)\} = G(s)F(s)$$

When you do this problem, you can simply look up the relevant Laplace transforms in our "Laplace Transform Table."

R3. Dumping Radioactive Waste Part I.

There is a pristine dumping ground totally free of any kind of trash so that the number of radioactive particles initially is $n(0) = 0$. Then in comes the dump trucks dumping radioactive waste according to the dumping function $d(t)$. The radioactive-decay differential equation

$$\frac{dn(t)}{dt} = -\lambda n(t)$$

must now be modified to include dumping.

$$\frac{dn(t)}{dt} = -\lambda n(t) + d(t)$$

Your first term on the right is your radioactive-loss due to decay. That piece is still proportional to $n(t)$. But you have to add the gain in radioactive particles $d(t)$ to get your total rate of change $\frac{dn(t)}{dt}$.

a) Show that the Laplace transform of your differential equation leads to your solution in the form $N(s) = F(s)D(s)$ where $L\{n(t)\} = N(s)$, $L\{d(t)\} = D(s)$, and $F(s)$ is everything else. What is $F(s)$?

b) Why is the function $f(t) = e^{-\lambda t}$?

R4. Dumping Radioactive Waste Part II. From what we learned in class, your result $N(s) = F(s)D(s)$ from R2 means that the solution $n(t)$ is given as the convolution

$$n(t) = d(t) * f(t) = \int_0^t d(u) f(t-u) du .$$

Consider a dumping function $d(t) = 1-t$ for $0 \le t \le 1$ and $d(t) = 0$ for $t \ge 1$. Note that $f(t) = e^{-\lambda t}$. Sketch this dumping function. Show that the "derivative trick" can be used to express your solution for this particular dumping function for times $t \ge 1$ as

$$n(t) = e^{-\lambda t}\left[1 - \frac{d}{d\lambda}\right]\left[\frac{1}{\lambda}(e^{\lambda} - 1)\right].$$

R5. Dumping Radioactive Waste Part III.

Show that your answer to R4 can be written as below. Note that $n(0) = 0$.

$$n(t) = e^{-\lambda(t-1)}\left[-\frac{e^{-\lambda}}{\lambda} + \frac{1}{\lambda^2}(1 - e^{-\lambda})\right] \text{ for } t \ge 1.$$

Extra Credit. Dumping Radioactive Waste Part IV.

Show that $n(t) = e^{-\lambda t}\left[\frac{e^{\lambda t} - 1}{\lambda} + \frac{1}{\lambda^2}(e^{\lambda t} - 1) - \frac{1}{\lambda} t e^{\lambda t}\right]$ for $0 \le t \le 1$. Then set t equal to 1 and apply your shifted radioactive decay factor $e^{-\lambda(t-1)}$ since there is no more dumping after t = 1 to show that $n(t) = n(1)e^{-\lambda(t-1)}$ gives you your answer to R5.