Theoretical Physics
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Chapter S Homework. Cauchy Integral Formula

Analytic Functions. Take $f(z) = z^n$, where $n \neq -1$ so we avoid $f(z) = \frac{1}{z}$. Your goal in this problem set is to show that $f(z) = z^n$ is analytic, i.e., the Cauchy-Riemann conditions hold. Then, we are in good shape since this covers functions $g(z) = \sum_{n=0}^{\infty} c_n z^n$ such as the cosine and sine.

S1. De Moivre's Formula or Theorem. Start with $z = x + iy = r\cos\theta + ir\sin\theta$, where $r^2 = x^2 + y^2$ and $\theta = \tan^{-1}\frac{y}{x}$. Show that $z^n = r^n(\cos n\theta + i\sin n\theta)$, which is De Moivre's Theorem when $r = 1$. Write out De Moivre's Theorem where there is an exponential on the left side of the equation and trigonometric functions on the right side. Draw a box around this important theorem.

Finally derive the following formulas, which will be needed in the next problem S2.

$$\frac{\partial r}{\partial x} = \cos\theta \qquad \frac{\partial r}{\partial y} = \sin\theta \qquad \frac{\partial \theta}{\partial x} = -\frac{\sin\theta}{r} \qquad \frac{\partial \theta}{\partial y} = \frac{\cos\theta}{r}$$

S2. Cauchy-Riemann Relations. Again start with $z = x + iy = r\cos\theta + ir\sin\theta$, where $r^2 = x^2 + y^2$ and $\theta = \tan^{-1}\frac{y}{x}$. For $f(z) = z^n = u + iv$, where n is not equal to -1, you know from the previous problem that $u = r^n\cos(n\theta)$ and $v = r^n\sin(n\theta)$. Show that the Cauchy-Riemann conditions are met for $f(z) = z^n = u + iv$ by using the chain rule to calculate your partial derivatives. As an example, here is one of the four partial derivatives from the Cauchy-Riemann relations.

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x}$$

For full credit, show the calculation of all partial derivatives from first principles with your results eventually expressed in terms of the polar coordinates r and θ. Finally show that the Cauchy-Riemann relations are satisfied.

Conclusion: Any function that can be expressed as a power series is analytic!