

**Theoretical Physics**  
**Prof. Ruiz, UNC Asheville, doctorphys on YouTube**  
**Chapter T Notes. Poles and the Residue Theorem**

**T1. Poles.**

First Review:

Complex Variable:  $z = x + iy$ , where  $\{x, y \in \mathbb{R}\}$

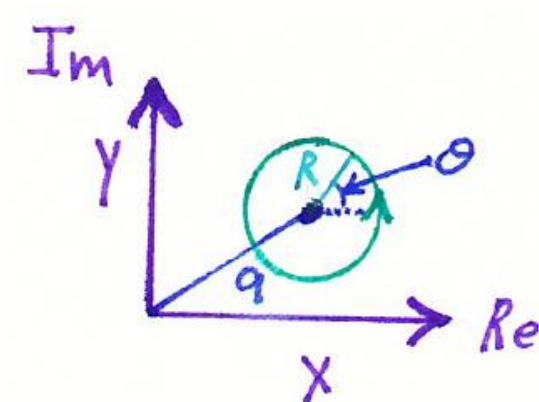
Complex Function:  $f(z) = u(x, y) + iv(x, y)$ , where  $\{u, v \in \mathbb{R}\}$

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$	The Cauchy-Riemann Conditions define analytic functions $f(z)$ , where
$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$	

$f'(z) = \frac{df(z)}{dz}$  is unambiguous and  $\oint_C f(z) dz = 0$ .

Integration is unambiguous and path independent.

Powers of  $z$  are analytic and therefore  $f(z) = \sum_{n=0}^{\infty} c_n z^n$  is analytic.



A pole  $a$  is a point where a non-analytic function "blows up." See the integrand below.

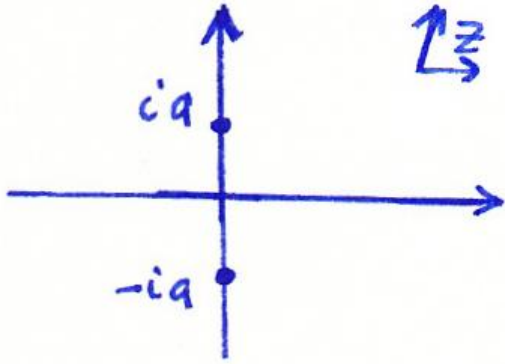
$$\oint \frac{1}{z - a} dz = 2\pi i$$

The Cauchy Integral Formula below involves an analytic function  $f(z)$  divided by  $z - a$ ,

which introduces a pole for the integrand at  $a$ .

$$\oint \frac{f(z)}{z - a} dz = 2\pi i f(a)$$

This is true for any closed path enclosing the pole.



Find the poles for  $f(z) = \frac{e^{imz}}{z^2 + a^2}$ .

The numerator is fine. The denominator though gives us two poles. We find them by factoring and setting the denominator to zero.

$$z^2 + a^2 = (z + ia)(z - ia) = 0$$

There are two solutions that make the denominator zero. These occur for

$$z = ia \text{ and } z = -ia .$$

These are the poles. These two poles are illustrated in the above figure.

**PT1 (Practice Problem).** Find the poles for the following function by factoring and check your answer using the quadratic formula.

$$f(z) = \frac{z^2 + 3}{z(z^2 + iz + 2)}$$

**PT2 (Practice Problem).** Find the poles for the following function using the quadratic formula.

$$f(z) = \frac{e^{ikz}}{z^2 - 6z + 25}$$

## T2. The Residue.

Note that for analytic functions  $\oint_C f(z) dz = 0$ .

For analytic functions  $f(z)$ , we have  $\oint \frac{f(z)}{z-a} dz = 2\pi i f(a)$ .

Now consider  $F(z) = \frac{f(z)}{z-a}$ , where  $f(z)$  is analytic. The function  $F(z)$  has a pole at  $z-a$ . We know

$$\oint F(z) dz = \oint \frac{f(z)}{z-a} dz = 2\pi i f(a)$$

We can get this answer by a)clearing out the denominator of  $F(z)$ , b)setting  $z = a$ , and c)multiplying by  $2\pi i$ .

$$\oint F(z) dz = 2\pi i \left[ (z-a)F(z) \right]_{z=a}$$

$$\oint F(z) dz = 2\pi i \left[ f(z) \right]_{z=a} = 2\pi i f(a)$$

The value  $f(a)$  is called the residue of  $F(z)$  at  $z = a$ .

$$f(a) = \text{Res}(F, a)$$

The residue of a simple pole is given by

$$\text{Res}(F, a) = \lim_{z \rightarrow a} \left[ (z-a)F(z) \right].$$

What about multiple poles? We take this up in our next section.

**T3. The Residue Theorem.** What about multiple poles?

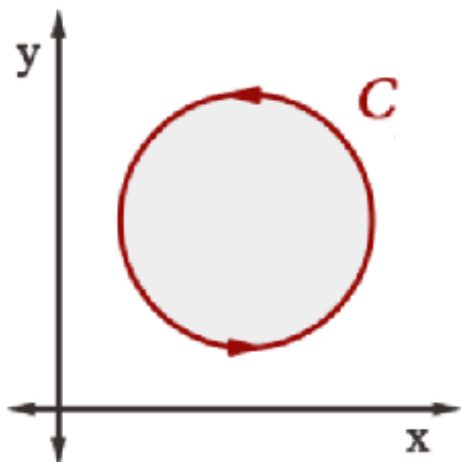


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**The Case of No Poles.**

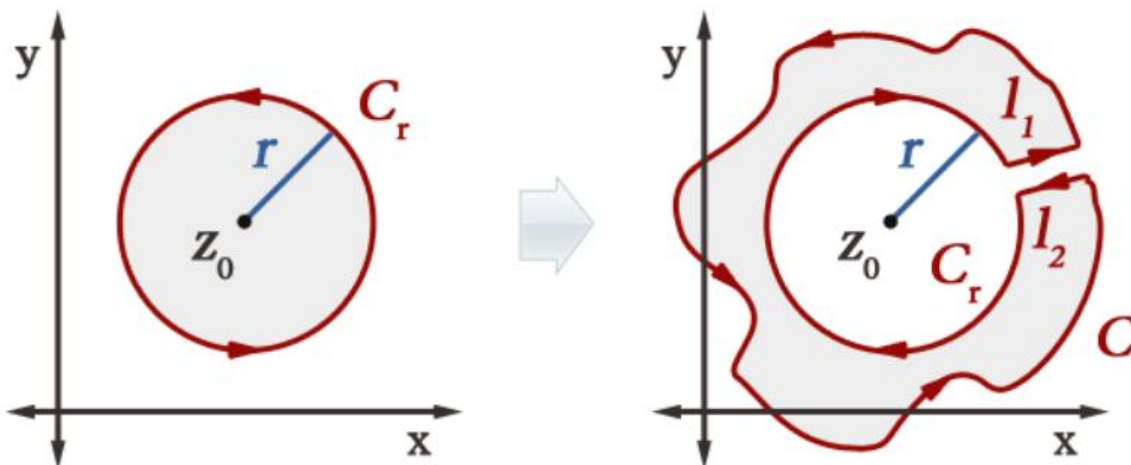
Well this is easy. A closed contour integral gives zero.

$$\oint_C F(z) dz = 0$$

**The Case of One Pole.** For the pole at  $z = z_0$ ,

$$\oint_C F(z) dz = 2\pi i \text{Res}(F, z_0)$$

We emphasize below that the closed integral can be of any shape!



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$$\oint_C F(z) dz + \int_{l_2} F(z) dz + \oint_{C_r} F(z) dz + \int_{l_1} F(z) dz = 0$$

The two line integrals in the limit as the gap approaches zero sum to zero.

$$\oint_C F(z) dz + \oint_{C_r} F(z) dz = 0$$

Since the second integral is clockwise, we get the following result.

$$\oint_C F(z) dz - 2\pi i \operatorname{Res}(F, z_0) = 0$$

$$\oint_C F(z) dz = 2\pi i \operatorname{Res}(F, z_0)$$

$$\oint_{\text{Any}} F(z) dz = 2\pi i \operatorname{Res}(F, z_0)$$

**The Case of Multiple Poles.** For multiple poles we apply a similar trick with paths.

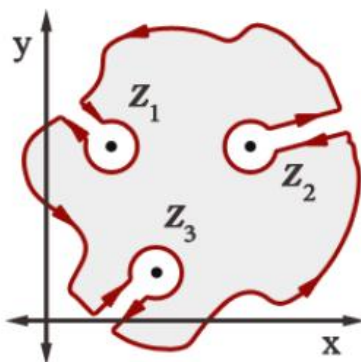


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$$\oint_C F(z) dz - 2\pi i \operatorname{Res}(F, z_1)$$

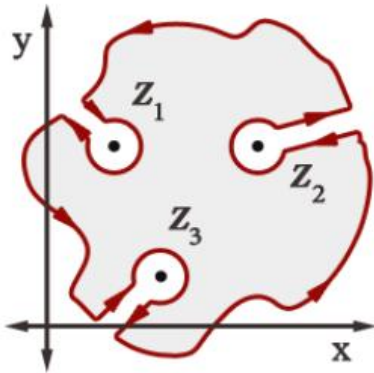
$$- 2\pi i \operatorname{Res}(F, z_2) - 2\pi i \operatorname{Res}(F, z_3) = 0$$

$$\boxed{\oint_C F(z) dz = 2\pi i \sum_n \operatorname{Res}(F, z_n)}$$

This is the Residue Theorem.

For our case with three poles:

$$F(z) = \frac{f(z)}{(z - z_1)(z - z_2)(z - z_3)}$$



$$\text{Res}(F, z_1) = \frac{f(z_1)}{(z_1 - z_2)(z_1 - z_3)}$$

$$\text{Res}(F, z_2) = \frac{f(z_2)}{(z_2 - z_1)(z_2 - z_3)}$$

$$\text{Res}(F, z_3) = \frac{f(z_3)}{(z_3 - z_1)(z_3 - z_2)}$$

#### T4. Complex Integration 1. $I = \int_{-\infty}^{\infty} \frac{dx}{1 + x^2}$

We will evaluate this integral using complex variable techniques. But first, we evaluate this integral from the observation that

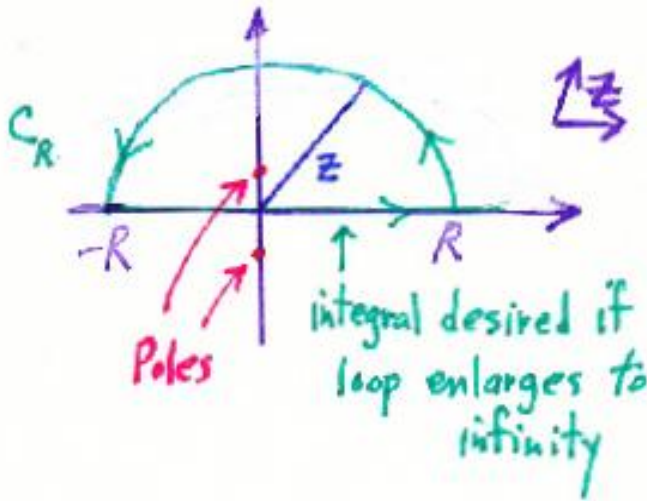
$$\frac{d}{dx} \tan^{-1} x = \frac{1}{1 + x^2}.$$

So we know the answer to this one. It is always good to try a new technique on something for which we know the answer.

$$I = \tan^{-1} x \Big|_{-\infty}^{\infty} = \frac{\pi}{2} - \left[ -\frac{\pi}{2} \right] = \pi$$

The method lies in this hope shown below as the radius  $R \rightarrow \infty$ .

$$\oint \frac{1}{1+z^2} dz = \int_{-\infty}^{\infty} \frac{dx}{1+x^2} + \int_{C_R} \frac{1}{1+z^2} dz = \int_{-\infty}^{\infty} \frac{dx}{1+x^2}$$



$$I = \oint \frac{1}{1+z^2} dz$$

$$\frac{1}{z^2+1} = \frac{1}{(z+i)(z-i)}$$

$$\text{Let } f(z) = \frac{1}{z+i}$$

$$I = \oint \frac{1}{(z+i)(z-i)} dz = \oint \frac{f(z)}{z-i} dz = 2\pi i f(i) = 2\pi i \frac{1}{2i} = \pi$$

$$\text{or } I = 2\pi i \text{Res}(F, i) = 2\pi i \left. \frac{1}{z+i} \right|_{z=i} = 2\pi i \frac{1}{2i} = \pi$$

All we have to do now is let  $R \rightarrow \infty$  and hope that the semicircle integration along

$C_R$  goes to zero. Then, the complete enclosed contour integral will give a nonvanishing answer for just the path along the complete x-axis and we are finished.

You know this must happen because you have your  $I = \pi$ , which you know is the answer. Show

$$\lim_{R \rightarrow \infty} I_{C_R} = 0, \text{ where } I_{C_R} = \int_{C_R} \frac{1}{1+z^2} dz.$$

$$z = Re^{i\theta} \text{ and } dz = iRe^{i\theta} d\theta$$

$$I_{C_R} = \int_{C_R} \frac{1}{1+(Re^{i\theta})^2} iRe^{i\theta} d\theta$$

$$I_{C_R} = \int_{C_R} \frac{1}{1+R^2 e^{2i\theta}} iRe^{i\theta} d\theta$$

For R large

$$I_{C_R} = \lim_{R \rightarrow \infty} \int_{C_R} \frac{1}{1+R^2 e^{2i\theta}} iRe^{i\theta} d\theta$$

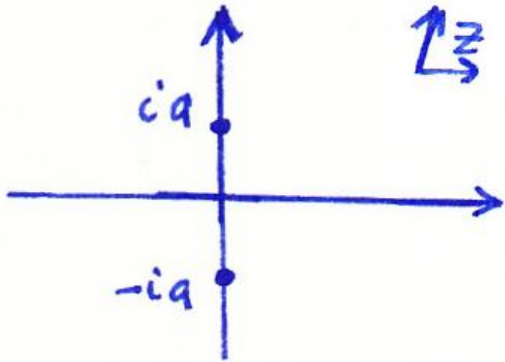
$$I_{C_R} = \lim_{R \rightarrow \infty} \int_{C_R} \frac{1}{R^2 e^{2i\theta}} iRe^{i\theta} d\theta$$

$$I_{C_R} = \lim_{R \rightarrow \infty} \int_{C_R} \frac{e^{-2i\theta}}{R^2} iRe^{i\theta} d\theta$$

$$I_{C_R} = i \lim_{R \rightarrow \infty} \int_{C_R} \frac{e^{-i\theta}}{R} d\theta = i \lim_{R \rightarrow \infty} \frac{1}{R} \int_{C_R} e^{-i\theta} d\theta = 0$$



**T5. Complex Integration 2.**  $I = \int_{-\infty}^{\infty} \frac{e^{imx} dx}{x^2 + a^2}$  with  $a > 0, m > 0$



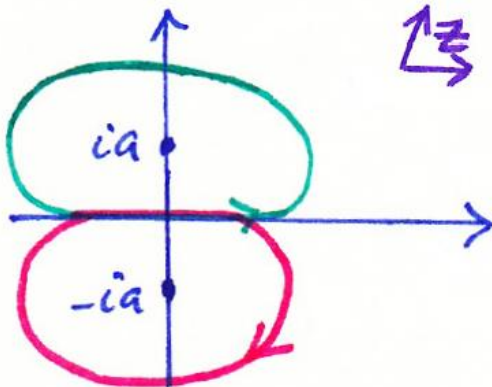
Let's organize our procedure into three steps.

**Step 1. Find Your Poles.**

$$F(z) = \frac{e^{imz}}{z^2 + a^2}$$

From  $z^2 + a^2 = (z + ia)(z - ia) = 0$  we find two poles.  $z_1 = ia$  and  $z_2 = -ia$ .

**Step 2. Know Where to Close (Choose Your Semicircle).** We need to choose the semicircle that will not mess up our vanishing semicircle result from the last section.



Which semicircle should we choose?  $m > 0$

Top - along upper imaginary axis

$$iR \Rightarrow e^{im(iR)} = e^{-mR}$$

Bottom - along lower imaginary axis

$$-iR \Rightarrow e^{im(-iR)} = e^{mR}$$

The top one is the one that will vanish.

**Step 3. Sum Your Residues (Use the Residue Theorem).** Note that we only have one pole inside our enclosed region of integration. This is  $z_1 = ia$ .

$$\oint_C F(z) dz = 2\pi i \text{Res}(F, ia) \quad \text{with} \quad F(z) = \frac{e^{imz}}{(z + ia)(z - ia)}$$

$$\text{Res}(F, ia) = \left. \frac{e^{imz}}{z + ia} \right|_{z=ia} = \frac{e^{-ma}}{2ia} \quad \text{and} \quad \int_{-\infty}^{\infty} \frac{e^{imx} dx}{x^2 + a^2} = \frac{\pi}{a} e^{-ma}$$

With the Real-Imaginary Trick, we now know the two following integrals where  $a > 0$  and  $m > 0$ .

$$I = \int_{-\infty}^{\infty} \frac{e^{imx} dx}{x^2 + a^2} = \int_{-\infty}^{\infty} \frac{\cos(mx) dx}{x^2 + a^2} + i \int_{-\infty}^{\infty} \frac{\sin(mx) dx}{x^2 + a^2}$$

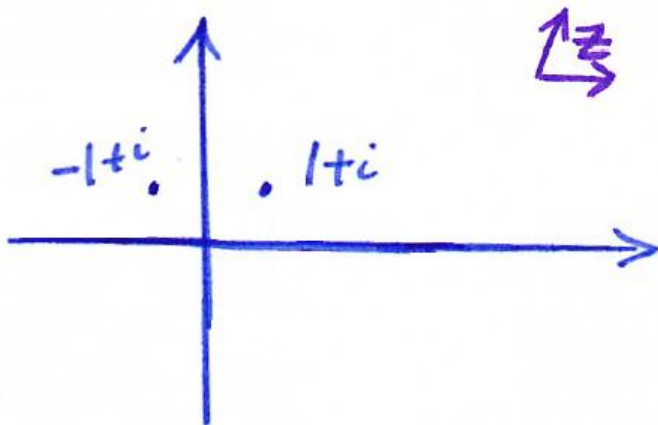
$$\int_{-\infty}^{\infty} \frac{\cos(mx) dx}{x^2 + a^2} = \frac{\pi}{a} e^{-ma} \quad \text{and} \quad \int_{-\infty}^{\infty} \frac{\sin(mx) dx}{x^2 + a^2} = 0$$

Note that also by the symmetry argument the second integral must be zero.

**T6. Complex Integration 3.**  $I = \int_{-\infty}^{\infty} \frac{e^{imx} dx}{x^2 - 2ix - 2}$  where  $m > 0$

**Step 1. Find Your Poles.**  $F(z) = \frac{e^{imz}}{z^2 - 2iz - 2}$

Use the quadratic formula with  $z^2 - 2iz - 2$  to get  $\frac{-(-2i) \pm \sqrt{(-2i)^2 - 4(1)(-2)}}{2(1)}$



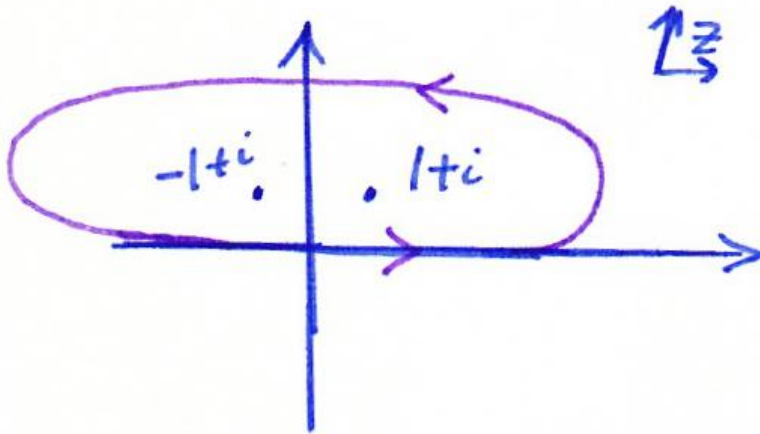
$$\frac{2i \pm \sqrt{-4+8}}{2} = \frac{2i \pm 2}{2}$$

Our poles are

$$z_1 = -1 + i$$

$$z_2 = 1 + i$$

**Step 2. Know Where to Close (Choose Your Semicircle).** We choose the upper plane since we have an exponential of the form  $e^{imz}$  and remember  $m > 0$ .



So we close in the upper plane so that we get

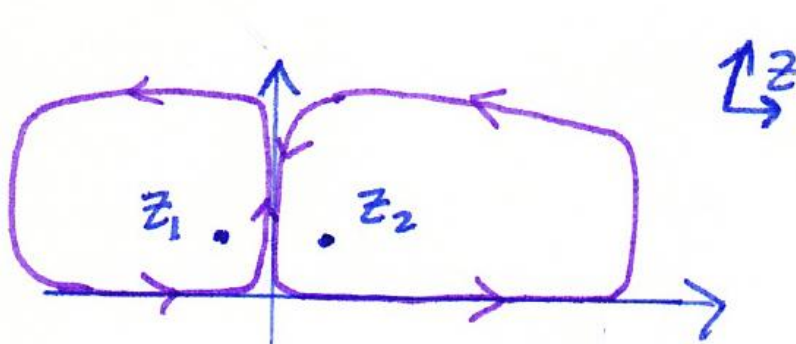
$$e^{im(iR)} = e^{-mR}$$

along the imaginary axis. As  $R \rightarrow \infty$  we get no trouble. Remember when

we showed for a  $\frac{1}{z^2}$  type of integrand that the semicircle path vanishes.

We just want to make sure here that the exponential factor does not mess us up.

**Step 3. Sum Your Residues (Use the Residue Theorem).**



The left figure is not needed. It just reminds you about the workings of the residue theorem. So we proceed to find the residues for the poles.

$$2\pi i \sum_n \text{Res}(F, z_n)$$

$$F(z) = \frac{e^{imz}}{(z - z_1)(z - z_2)} \quad \text{where } z_1 = -1+i \text{ and } z_2 = 1+i$$

$$\sum_n \text{Res}(F, z_n) = \left. \frac{e^{imz}}{z - z_2} \right|_{z=z_1} + \left. \frac{e^{imz}}{z - z_1} \right|_{z=z_2}$$

$$\sum_n Res(F, z_n) = \frac{e^{imz_1}}{z_1 - z_2} + \frac{e^{imz_2}}{z_2 - z_1} \text{ where } z_1 = -1+i \text{ and } z_2 = 1+i$$

$$I = 2\pi i \sum_n Res(F, z_n)$$

$$I = \frac{2\pi i}{z_1 - z_2} [e^{imz_1} - e^{imz_2}] = \frac{2\pi i}{(-2)} [e^{imz_1} - e^{imz_2}]$$

$$I = -\pi i [e^{im(-1+i)} - e^{im(1+i)}]$$

$$I = -\pi i [e^{m(-i-1)} - e^{m(i-1)}]$$

$$I = -\pi i [e^{-im} e^{-m} - e^{im} e^{-m}]$$

$$I = -\pi i e^{-m} [e^{-im} - e^{im}]$$

$$I = \pi i e^{-m} [e^{im} - e^{-im}]$$

$$I = \pi i e^{-m} (2i) \left[ \frac{e^{im} - e^{-im}}{2i} \right]$$

$$I = \int_{-\infty}^{\infty} \frac{e^{imx} dx}{x^2 - 2ix - 2} = -2\pi e^{-m} \sin m$$