Theoretical Physics
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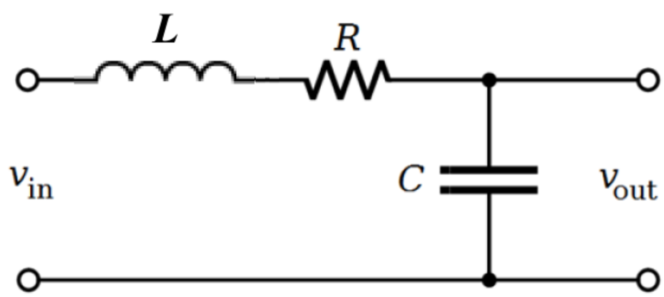
Chapter V Homework. Transfer Functions

V1. The LRC Circuit and Phasors. Use phasors to find the transfer function $H(\omega) = \frac{v_{out}(t)}{v_{in}(t)} = \frac{i(t)Z_C}{i(t)Z_{LRC}}$ for the LRC circuit shown below, where $i(t)$ is the current in the single loop circuit, Z_C is the impedance of the capacitor, and Z_{LRC} is the total impedance in the circuit. Your impedances for the individual circuit elements are $Z_L = j\omega L$, $Z_R = R$, and $Z_C = \frac{1}{j\omega C}$, where $j = \sqrt{-1}$. Note that L, R, and C are in series so that the impedances in this case add.



After you find your transfer function $H(\omega)$, derive the following results.

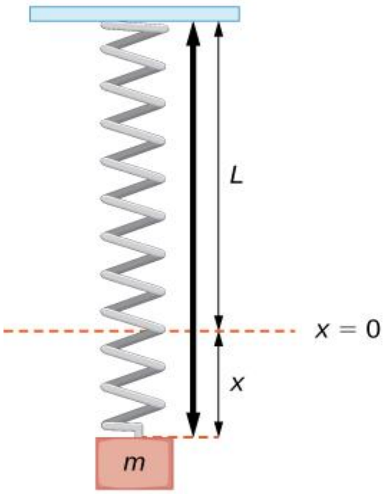
$$|H(\omega)| = \frac{\omega_0^2}{\sqrt{(\omega^2 - \omega_0^2)^2 + 4\beta^2\omega^2}}, \text{ where } \omega_0 = \frac{1}{\sqrt{LC}} \text{ and } \beta = \frac{R}{2L}.$$

V2. The LRC Circuit and Fourier Transform. The voltage drops in the above circuit are a) resistor: $V_R = iR = \frac{dq}{dt}R$, b) capacitor: $V_C = \frac{q}{C}$, c) inductor: $V_L = L\frac{di}{dt} = L\frac{d^2q}{dt^2}$.
Therefore, using physics for the circuit in V1 without phasors, you have the following differential equations.

$$v_{in}(t) = L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} \text{ with } v_{out}(t) = \frac{q}{C}$$

Take the Fourier transform and show that your result for $H(\omega)$ is the same as that found in V1.

V3. Mechanical Resonance.



In introductory physics you learn about Hooke's Law.

$$F_{spring} = -kx$$

If we immerse the system in a fluid, we can approximate an additional damping force as proportional to the speed of the mass.

$$F_{fluid} = -b\frac{dx}{dt}$$

We can also wiggle the bar at the top to drive the system with an applied sinusoidal force.

$$F_{applied} = F_0 \sin \omega t = F(t)$$

Putting all this together and applying Newton's Law F = ma, we get

$$-kx - b\frac{dx}{dt} + F(t) = m\frac{d^2x}{dt^2} \, ,$$

which can be written as

$$F(t) = m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx$$

This differential has the same form as the one in V2 for the LRC circuit. Note that x is analogous to the charge q. Your voltage $v_{out}(t) = v_C(t) = q/C$. Show by analogy, that the amplitude of response for x can be written as

$$A(\omega) = \frac{\omega_0^2}{\sqrt{(\omega^2 - \omega_0^2)^2 + 4\beta^2\omega^2}}$$ and find ω_0 and β in terms of m , b , k , and F_0 .

Note that you might need all of the parameters m , b , k , and F_0 for your ω_0 and β .

V4. The Resonance Frequency. From the response formula

$$A(\omega) = \frac{\omega_0^2}{\sqrt{(\omega^2 - \omega_0^2)^2 + 4\beta^2\omega^2}}$$

For both the driven electrical and driven mechanical systems, show that the maximum amplitude response occurs at $\omega_R = \sqrt{\omega_0^2 - 2\beta^2}$, which is called the resonance frequency. The best method to use here is your standard max-min problem by setting $\frac{dA(\omega)}{d\omega} = 0$. But let me tell you about a short-cut trick since this derivative is messy. The shortcut works since there is no ω in the numerator. In this case, you can set what is inside the square root of the denominator to zero since when your denominator is at a minimum, you have a maximum for your amplitude parameter.

So you solve for the maximum response of the amplitude of oscillation or the charge on the capacitor.

V5. The Frequency for Maximum Current or Velocity Amplitude. Find the maximum for the current amplitude (electrical case) or velocity amplitude for the oscillation (mechanical case). It will NOT occur at ω_R . Hint for Solution: The voltage across the resistor is in sync with the input voltage and your voltage across the resistor is proportional to the current since $V_R(t) = i(t)R$. So redo the transfer function in V1 by taking the voltage across the resistor as your output. You can then find the max for the current amplitude directly that way much faster. Make sure you keep all the ω terms in the denominator when finding $H(\omega)$ and $|H(\omega)|$. Then, you will not have to take a derivative since the answer can be obtained by inspection.