Theoretical Physics
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Chapter W Homework. Principle of Least Action

W1. Abram Bader, Feynman's High School Physics Teacher. "Bader asked Feynman to consider a less intuitive quantity than the sum of these energies: their difference. Subtracting the potential energy from the kinetic energy was as easy as adding them. It was just a matter of changing signs. But understanding the physical meaning was harder. Far from being conserved, this quantity the - *action* Bader said - changed constantly. Bader had Feynman calculate it for the ball's entire flight to the window. And he pointed out what seemed to Feynman a miracle. At any particular moment the action might rise or fall, but when the ball arrived at its destination, the path it had followed would always be the path for which the total action was least. For any other path Feynman might try drawing on the blackboard - a straight line from the ground to the window, a higher-arcing trajectory, or a trajectory that deviated however slightly from the fated path - he would find a greater average difference between kinetic and potential energy." ***James Gleick, The Life and Science of Richard Feynman (Vintage Books, New York. 1993)***

Consider a trajectory upward to the window. The equations from introductory physics class are given below. Note that the velocity is the derivative of the x(t) function.

$$x = x_0 + v_0 t - \frac{1}{2} g t^2 \qquad v = v_0 - gt$$

The Lagrangian is $L = \frac{1}{2} m v^2 - mgx$.

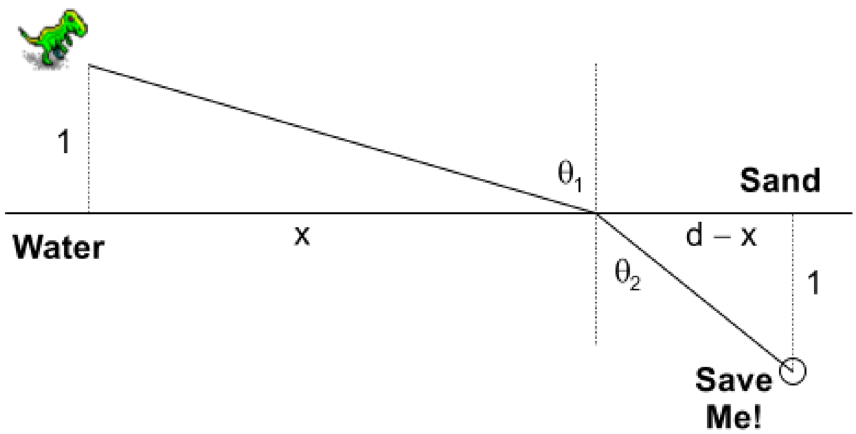
You are going to integrate the action integral $S = \int_0^1 L\, dt$ for several paths.

Take $g = 2$ and $m = 1$ so that $mgx = 2x$. Evaluate the action integral S for the correct path (a) which is of the form $x = x_0 + v_0 t - \frac{1}{2} g t^2$ and the 3 artificial paths (b), (c), and (d).

(a) $x(t) = 2t - t^2$, (b) $x(t) = t$, (c) $x(t) = t^2$, (d) $x(t) = t^3$.

List these paths in order from minimum total action to greatest total action for the trips.

W2. The Principle of Least Time. The *Principle of Least Time,* formulated by Fermat (1601-1665), states that light gets from one point to another taking the least amount of time. This principle is illustrated below by the lifeguard problem. Remember that Dino can run faster than he can swim.



Do the usual max-min type of problem from calculus class where here you minimize the time to go from the point marked by Dino to "Save Me." Take the speed in medium 1 (top medium) to be v_1 and the speed in medium 2 (bottom medium) to be v_2. Note the "x" and "d–x" to start you off. Show that the max-min problem leads to

$$n_1 \sin\theta_1 = n_2 \sin\theta_2 ,$$

where $n_1 = c / v_1$ and $n_2 = c / v_2$, with c being a constant. This is Snell's Law for light where n is called the index of refraction. Now sketch a few thin layers where the index of refraction for each layer is n_1, n_2, n_3, ... and light travels distances s_1, s_2, s_3, ... in the respective layers. Show that the time for light to travel through these layers is given by $t = \frac{1}{c}\sum n_i s_i$. From this, show that for a continuously changing medium such as infinitesimal air layers getting cooler and cooler, the time is given by

$$t = \frac{1}{c}\int n(s)\,ds .$$

The principle of least time dictates that the above integral is a minimum.