**Theoretical Physics** **Exam 2** **Spring 2020**
**Prof. Ruiz, UNC Asheville** **Study Guide**

## PART I: MEMORIZATION. Memorize the Following Equations.
## Some equations come from earlier courses and some can be reasoned out.

### 1. Class G. Ideal Gas Law and Thermodynamics.

$PV = nRT$ and $PV = NkT$ with $\frac{1}{2}m\overline{v^2} = \frac{3}{2}kT$ .

Discrete : $\overline{v^2} = \frac{N_1}{N}v_1^2 + \frac{N_2}{N}v_2^2 + \frac{N_3}{N}v_3^2 + \ldots$ and $\frac{N_1}{N} + \frac{N_2}{N} + \frac{N_3}{N} + \ldots = 1$

Continuous: $\overline{v^2} = \int_0^\infty v^2 f(v)dv$ and $\int_0^\infty f(v)dv = 1$

$$U = N\frac{1}{2}m\overline{v^2} = \frac{3}{2}NkT = \frac{3}{2}nRT$$

$\Delta U = \Delta Q - \Delta W$ where $\Delta W = P\Delta V$ and $W_{1\to 2} = \int_{V_1}^{V_2} PdV$

$$c_V \equiv \frac{1}{n}\frac{\Delta Q}{\Delta T}\bigg|_V = \frac{1}{n}\frac{\Delta Q}{\Delta T} = \frac{1}{n}\frac{3}{2}\frac{nR\Delta T}{\Delta T} = \frac{3}{2}R \qquad c_P \equiv \frac{1}{n}\frac{\Delta Q}{\Delta T}\bigg|_P$$

1. Isometric, Isochoric (constant V): $\Delta V = 0$ . $W = \int P\,dV = 0$

2. Isobaric (const P): $\Delta P = 0$ . $W = \int_{V_1}^{V_2} P\,dV = P\int_{V_1}^{V_2} dV = P(V_2 - V_1)$

3. Isothermal (const T): $\Delta T = 0$ .

$$W = \int_{V_1}^{V_2} P\,dV = nRT\int_{V_1}^{V_2} \frac{1}{V}dV = nRT\ln\frac{V_2}{V_1}$$

4. Adiabatic (no heat flow): $\Delta Q = 0$ . $P_1V_1^\gamma = P_2V_2^\gamma$ , i.e., $PV^\gamma = const$

## Class H. Statistical Mechanics.

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{1\cdot 2}a^{n-2}b^2 ...+ c(n,r)a^{n-r}b^r ...+ b^n$$

Partition Function: $Z = \sum_i e^{-\beta\varepsilon_i} = e^{-\beta\varepsilon_1} + e^{-\beta\varepsilon_2} + \ldots$ with $\beta = \frac{1}{kT}$.

Occupation Number: $n_i = N\frac{e^{-\beta\varepsilon_i}}{Z}$ for the i^th^ energy level $\varepsilon_i$.

Entropy: $S = k\ln\Omega$ where $\Omega$ is the number of microstates for the system.

Entropy: $\Delta S = \Delta Q / T$ macrostate variable. Total entropy increases.

All good macroscopic variables: $\Delta U = T\Delta S - P\Delta V$

## I1. Quantization.

Photon Energy: $E_{photon} = hf$

Wave Relation: $c = \lambda f$

de Broglie Relation: $\lambda = \frac{h}{p}$

Wave Number: $k = \frac{2\pi}{\lambda}$

Angular Frequency: $\omega = 2\pi f$

Classical Energy: $\frac{p^2}{2m} + V = E$, where the momentum is $p = mv$.

**J. Spinors.** Given $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

1. The Trace: $Tr(A) = a + d$

2. The Transpose: $A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$. Note that $Tr(A^T) = Tr(A)$.

3. The Complex Conjugate: $A^* = \begin{bmatrix} a^* & b^* \\ c^* & d^* \end{bmatrix}$.

4. The Hermitian Conjugate: $A^\dagger = \begin{bmatrix} a^* & c^* \\ b^* & d^* \end{bmatrix} = (A^T)^* = (A^*)^T$ .

5. The Determinant: $\det(A) \equiv |A| = ad - cb$ .

6. The Inverse: $A^{-1}$ such that $AA^{-1} = I$ , where $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Kronecker Delta Symbol: $\delta_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$

Levi-Civita or Permutation Symbol:

$$\varepsilon_{ijk} = \begin{cases} +1 & \text{if } (i,j,k) \text{ is } (1,2,3), (3,1,2) \text{ or } (2,3,1), \\ -1 & \text{if } (i,j,k) \text{ is } (1,3,2), (3,2,1) \text{ or } (2,1,3), \\ 0 & \text{if } i = j \text{ or } j = k \text{ or } k = i \end{cases}$$

Commutator: $[A, B] \equiv AB - BA$

Anticommutator: $\{A, B\} \equiv AB + BA$

**K. The Pauli Equation.**

Angular Momentum:

$$\vec{L} = \vec{r} \times \vec{p} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ p_x & p_y & p_z \end{vmatrix}$$

**L. The Dirac Equation.** Nothing to memorize.

**M. The Method of Frobenius.** $y(x) = \sum_{k=0}^{\infty} a_k x^k$ and the steps for using it.

**N. The Dirac Delta Function.**

$$\delta(x) = \begin{cases} 0, & x \neq 0 \\ +\infty, & x = 0 \end{cases}$$

with $\int_{-\infty}^{\infty} \delta(x)\, dx = 1$

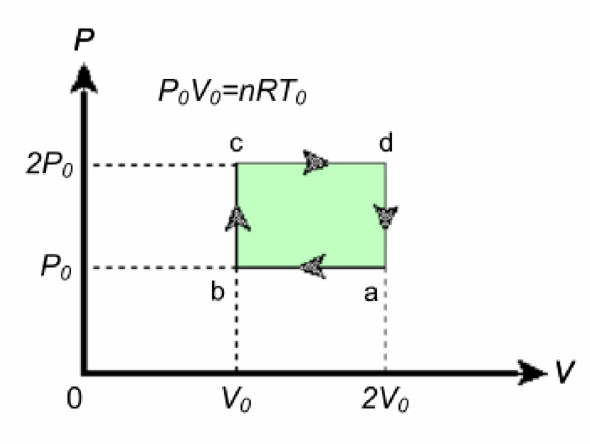
$$\int_{-\infty}^{\infty} \delta(x) f(x) dx = f(0)$$ and

$$\int_{-\infty}^{\infty} \delta(x-a) f(x) dx = f(a)$$

## PART II: Working Problems.

**1. Engine Cycles and the PV Diagram for Ideal Gas.** Best practice is to list the changes of energy, heat, and work for each part of the cycle, and finally calculate the efficiency.

**a) Rectangular Cycle (Net Work Done is Enclosed Green Region):** Area $= P_0V_0 = nRT_0$




Compression Stroke: a to b
(compressing the gas fuel)

Ignition Phase: b to c
(with sudden pressure increase)

Expansion Phase: c to d
(engine does useful work)

Pressure-Drop Phase: d to a
(returning to the initial PV point)

Can you determine that $T_b = T_0$ , $T_a = 2T_0$ , $T_c = 2T_0$ , $T_d = 4T_0$ from the ideal gas law?

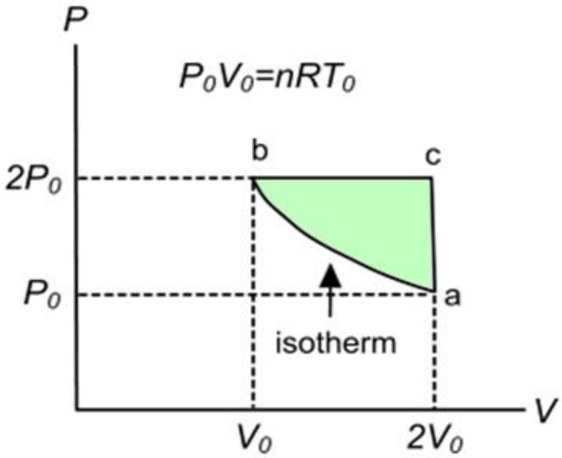
Can you fill in the following table using $\Delta U = \Delta Q - P\Delta V$ and $U = \frac{3}{2}nRT$ ?

| | $\Delta U$ | $\Delta Q$ | $\Delta W$ |
|---|---|---|---|
| a to b | $-\frac{3}{2}nRT_0$ | $-\frac{5}{2}nRT_0$ | $-nRT_0$ |
| b to c | $\frac{3}{2}nRT_0$ | $\frac{3}{2}nRT_0$ | $0$ |
| c to d | $3nRT_0$ | $5nRT_0$ | $2nRT_0$ |
| d to a | $-3nRT_0$ | $-3nRT_0$ | $0$ |

Efficiency: $\eta \equiv \frac{W_{net}}{Q_{in}} = \frac{nRT_0}{\frac{3}{2}nRT_0 + 5nrT_0} = \frac{nRT_0}{\left[\frac{3}{2}+\frac{10}{2}\right]nRT_0} = \frac{1}{13/2} = \frac{2}{13}$.

**1st Check:** The sum of the column $\Delta U = 0$. **2nd Check:** The sum of the column $\Delta Q = nRT_0$ must be equal to the sum of column $\Delta W = nRT_0$ so for the sums: $\Delta U = \Delta Q - \Delta W = 0$.

**b) Three-Part Cycle with Isotherm**




Description of the Cycle.

a-b (the isothermal compression),

b-c (the isobaric expansion), and

c-a (the isometric pressure drop).

Can you determine all the entries below and then calculate the efficiency?

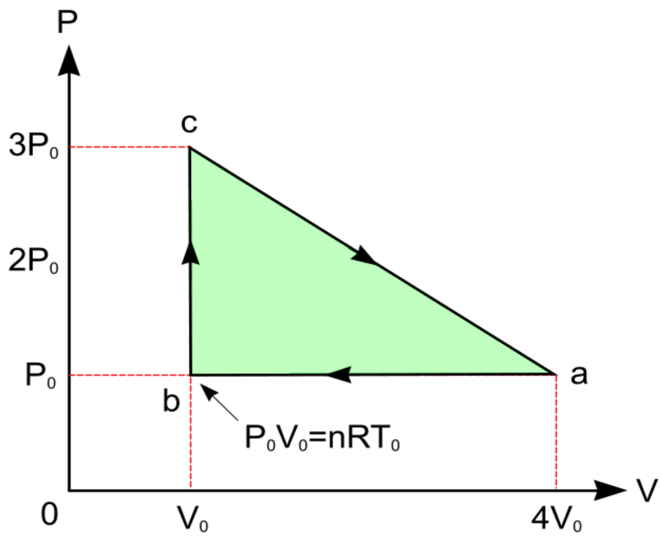
| | $\Delta U$ | $\Delta Q$ | $\Delta W$ |
|---|---|---|---|
| a to b | $0$ | $-2nRT_0 \ln 2$ | $-2nRT_0 \ln 2$ |
| b to c | $3nRT_0$ | $5nRT_0$ | $2nRT_0$ |
| c to a | $-3nRT_0$ | $-3nRT_0$ | $0$ |

Efficiency: $\eta \equiv \dfrac{W_{net}}{Q_{in}} = \dfrac{2nRT_0 - 2nrT_0 \ln 2}{5nrT_0} = \dfrac{(2 - 2\ln 2)RT_0}{5nRT_0} = \dfrac{2(1 - \ln 2)}{5}$ .

**1st Check:** The sum of the column $\Delta U = 0$.

**2nd Check:** The sum of the column $\Delta Q = 2nRT_0 - 2nRT_0 \ln 2$ must be equal to the sum of column $\Delta W = 2nRT_0 - 2nRT_0 \ln 2$ so for the sums: $\Delta U = \Delta Q - \Delta W = 0$.

**c) A Triangular Cycle.** Can you complete the table below and calculate the efficiency?




The net work done is the green area, the area of a triangle.

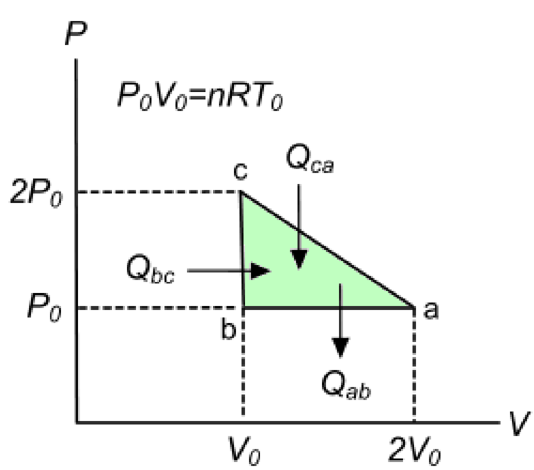
$$W_{net} = \frac{1}{2}(3P_0 - P_0)(4V_0 - V_0)$$

$$W_{net} = \frac{1}{2}(2P_0)(3V_0) = 3P_0V_0 = 3nRT_0$$

| | $\Delta U$ | $\Delta Q$ | $\Delta W$ |
|---|---|---|---|
| a to b | $-4.5nRT_0$ | $-7.5nRT_0$ | $-3nRT_0$ |
| b to c | $3nRT_0$ | $3nRT_0$ | $0$ |
| c to a | $1.5nRT_0$ | $7.5nRT_0$ | $6nRT_0$ |

$$\eta \equiv \frac{W_{net}}{Q_{in}} = \frac{3nRT_0}{3nrT_0 + 7.5nRT_0} = \frac{3}{10.5} = \frac{3}{21/2} = \frac{6}{21} = \frac{2}{7}$$

**d) A Second Triangular Cycle**




Can you show that the efficiency below is correct?

$$\eta \equiv \frac{W_{net}}{Q_{in}} = \frac{\frac{1}{2}nRT_0}{\frac{3}{2}nRT_0 + \frac{3}{2}nRT_0} = \frac{1/2}{3} = \frac{1}{6}$$

## E2. Statistical Mechanics.

**1, Two Quantum States (Symmetric About Zero Energy).** There are two energy states $\varepsilon_1 = -\varepsilon < 0$ and $\varepsilon_2 = +\varepsilon > 0$. Find the average energy as a function of temperature T. What happens at absolute zero temperature and when the temperature goes to infinity?

$$\varepsilon_2 = +\varepsilon \quad \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_}$$

$$0 \quad \text{- - - - - - - - - - - - - - - - - - - -}$$

$$\varepsilon_1 = -\varepsilon \quad \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_}$$

$$Z = \sum_i e^{-\frac{\varepsilon_i}{kT}} = e^{-\frac{\varepsilon_1}{kT}} + e^{-\frac{\varepsilon_2}{kT}} = e^{\frac{\varepsilon}{kT}} + e^{-\frac{\varepsilon}{kT}}$$

$$\frac{n_1}{N} = \frac{e^{\frac{\varepsilon}{kT}}}{Z} \quad \text{and} \quad \frac{n_2}{N} = \frac{e^{-\frac{\varepsilon}{kT}}}{Z}$$

$$\overline{E} = \frac{n_1}{N}(-\varepsilon) + \frac{n_2}{N}\varepsilon$$

$$\overline{E} = -\varepsilon\left[\frac{e^{\frac{\varepsilon}{kT}} - e^{-\frac{\varepsilon}{kT}}}{Z}\right] = -\varepsilon\left[\frac{e^{\frac{\varepsilon}{kT}} - e^{-\frac{\varepsilon}{kT}}}{e^{\frac{\varepsilon}{kT}} + e^{-\frac{\varepsilon}{kT}}}\right] = -\varepsilon \tanh\frac{\varepsilon}{kT}$$

$$\lim_{T\to 0}\overline{E} = -\varepsilon\lim_{T\to 0}\left[\frac{e^{\frac{\varepsilon}{kT}} - e^{-\frac{\varepsilon}{kT}}}{e^{\frac{\varepsilon}{kT}} + e^{-\frac{\varepsilon}{kT}}}\right] = -\varepsilon\lim_{T\to 0}\left[\frac{e^{\frac{\varepsilon}{kT}}}{e^{\frac{\varepsilon}{kT}}}\right] = -\varepsilon\lim_{T\to 0}1 = -\varepsilon.$$

All particles are in the lower energy level, the ground state.

$$\lim_{T\to\infty}\overline{E} = -\varepsilon\lim_{T\to\infty}\left[\frac{e^{\frac{\varepsilon}{kT}} - e^{-\frac{\varepsilon}{kT}}}{e^{\frac{\varepsilon}{kT}} + e^{-\frac{\varepsilon}{kT}}}\right] = -\varepsilon\left[\frac{1-1}{1+1}\right] = 0$$

Half of the particles are in the lower energy level and half are in the upper energy level.

**2, Two Quantum States (With Ground State Energy Referenced at Zero).** There are two energy states $\varepsilon_1 = 0$ and $\varepsilon_2 = \varepsilon > 0$. Find the average energy as a function of temperature T. What happens at absolute zero temperature and when the temperature goes to infinity?

$$\varepsilon_2 = +\varepsilon \quad \underline{\hspace{6cm}}$$

$$\varepsilon_1 = 0 \quad \underline{\hspace{6cm}}$$

$$Z = \sum_i e^{-\frac{\varepsilon_i}{kT}} = e^{-\frac{\varepsilon_1}{kT}} + e^{-\frac{\varepsilon_2}{kT}} = e^0 + e^{-\frac{\varepsilon}{kT}} = 1 + e^{-\frac{\varepsilon}{kT}}$$

$$\frac{n_1}{N} = \frac{1}{Z} \text{ and } \frac{n_2}{N} = \frac{e^{-\frac{\varepsilon}{kT}}}{Z}$$

$$\overline{E} = \frac{n_1}{N}(0) + \frac{n_2}{N}\varepsilon = \frac{n_2}{N}\varepsilon$$

$$\overline{E} = \varepsilon\left[\frac{e^{-\frac{\varepsilon}{kT}}}{Z}\right] = \varepsilon\left[\frac{e^{-\frac{\varepsilon}{kT}}}{1 + e^{-\frac{\varepsilon}{kT}}}\right]$$

$$\lim_{T\to 0}\overline{E} = \varepsilon \lim_{T\to 0}\left[\frac{e^{-\frac{\varepsilon}{kT}}}{1 + e^{-\frac{\varepsilon}{kT}}}\right] = 0 .$$

All particles are in the lower energy level, the ground state.

$$\lim_{T\to\infty}\overline{E} = \varepsilon \lim_{T\to\infty}\left[\frac{e^{-\frac{\varepsilon}{kT}}}{1 + e^{-\frac{\varepsilon}{kT}}}\right] = \varepsilon\left[\frac{1}{1+1}\right] = \frac{\varepsilon}{2}$$

Half of the particles are in the lower energy level and half are in the upper energy level.

3. **Four Quantum States (Symmetric About Zero Energy).** There are four energy states.

$$\varepsilon_2 = +2\varepsilon \quad \rule{6cm}{0.4pt}$$

$$\varepsilon_2 = +\varepsilon \quad \rule{6cm}{0.4pt}$$

$$0 \quad \text{- - - - - - - - - - - - - - - - - - - - - - - - - -}$$

$$\varepsilon_1 = -\varepsilon \quad \rule{6cm}{0.4pt}$$

$$\varepsilon_1 = -2\varepsilon \quad \rule{6cm}{0.4pt}$$

Where are the particles at absolute zero temperature and when the temperature goes to infinity?

$$Z = \sum_i e^{-\frac{\varepsilon_i}{kT}} = e^{-\frac{\varepsilon_1}{kT}} + e^{-\frac{\varepsilon_2}{kT}} + e^{-\frac{\varepsilon_3}{kT}} + e^{-\frac{\varepsilon_4}{kT}} = e^{\frac{2\varepsilon}{kT}} + e^{\frac{\varepsilon}{kT}} + e^{-\frac{\varepsilon}{kT}} + e^{-\frac{2\varepsilon}{kT}}$$

$$\frac{n_1}{N} = \frac{e^{\frac{2\varepsilon}{kT}}}{Z} \qquad \frac{n_2}{N} = \frac{e^{\frac{\varepsilon}{kT}}}{Z} \qquad \frac{n_3}{N} = \frac{e^{-\frac{\varepsilon}{kT}}}{Z} \qquad \frac{n_4}{N} = \frac{e^{-\frac{2\varepsilon}{kT}}}{Z}$$

Absolute Zero, T = 0.

$$\frac{n_1}{N} = \lim_{T\to 0} \frac{e^{\frac{2\varepsilon}{kT}}}{e^{\frac{2\varepsilon}{kT}} + e^{\frac{\varepsilon}{kT}} + e^{-\frac{\varepsilon}{kT}} + e^{-\frac{2\varepsilon}{kT}}} = \lim_{T\to 0} \frac{1}{1 + e^{-\frac{\varepsilon}{kT}} + e^{-\frac{3\varepsilon}{kT}} + e^{-\frac{4\varepsilon}{kT}}} = 1$$

Since $\lim_{x\to 0} e^{-\frac{1}{x}} = \lim_{y\to\infty} e^{-y} = 0$ .

$$\frac{n_2}{N} = \lim_{T\to 0} \frac{e^{\frac{\varepsilon}{kT}}}{e^{\frac{2\varepsilon}{kT}} + e^{\frac{\varepsilon}{kT}} + e^{-\frac{\varepsilon}{kT}} + e^{-\frac{2\varepsilon}{kT}}} = \lim_{T\to 0} \frac{1}{e^{\frac{\varepsilon}{kT}} + 1 + e^{-\frac{2\varepsilon}{kT}} + e^{-\frac{3\varepsilon}{kT}}} = 0$$

Similar for the two upper states. All particles are in the ground state.

Super High Temperature, T to Infinity.

$$\frac{n_1}{N} = \lim_{T\to\infty} \frac{e^{\frac{2\varepsilon}{kT}}}{e^{\frac{2\varepsilon}{kT}} + e^{\frac{\varepsilon}{kT}} + e^{-\frac{\varepsilon}{kT}} + e^{-\frac{2\varepsilon}{kT}}} = \frac{1}{4}$$

Since $\lim\limits_{x\to\infty} e^{\pm\frac{1}{x}} = \lim\limits_{y\to 0} e^{\pm y} = 1$. Similarly $\frac{n_2}{N} = \frac{n_3}{N} = \frac{n_4}{N} = \frac{1}{4}$

The particles are equally distributed, the most disorganized state, the greatest entropy.

4. **Some Different States with Same Energy.** Find the average energy. What happens at absolute zero and at super high temperatures?

Consider $\varepsilon_1 = 0$, $\varepsilon_2 = 0$, $\varepsilon_3 = \varepsilon$, $\varepsilon_4 = \varepsilon$, $\varepsilon_5 = 2\varepsilon$, where $\varepsilon > 0$.

$$Z = \sum_i e^{-\frac{\varepsilon_i}{kT}} = e^0 + e^0 + e^{-\frac{\varepsilon}{kT}} + e^{-\frac{\varepsilon}{kT}} + e^{-\frac{2\varepsilon}{kT}} = 2 + 2e^{-\frac{\varepsilon}{kT}} + e^{-\frac{2\varepsilon}{kT}}$$

$$\overline{E} = \frac{n_1}{N}\cdot 0 + \frac{n_2}{N}\cdot 0 + \frac{n_3}{N}\varepsilon + \frac{n_4}{N}\varepsilon + \frac{n_5}{N}2\varepsilon \qquad \overline{E} = \frac{\varepsilon e^{-\frac{\varepsilon}{kT}} + \varepsilon e^{-\frac{\varepsilon}{kT}} + 2\varepsilon e^{-\frac{2\varepsilon}{kT}}}{2 + 2e^{-\frac{\varepsilon}{kT}} + e^{-\frac{2\varepsilon}{kT}}}$$

Absolute Zero, T = 0. $\frac{n_1}{N} = \lim\limits_{T\to 0} \frac{e^0}{2 + 2e^{-\frac{\varepsilon}{kT}} + e^{-\frac{2\varepsilon}{kT}}} = \frac{1}{2}$ $\quad \frac{n_2}{N} = \frac{1}{2}$

$$\frac{n_3}{N} = \lim_{T\to 0} \frac{e^{-\frac{\varepsilon}{kT}}}{2 + 2e^{-\frac{\varepsilon}{kT}} + e^{-\frac{2\varepsilon}{kT}}} = 0 \qquad \frac{n_4}{N} = 0 \qquad \frac{n_5}{N} = 0 \qquad \overline{E} = 0$$

The particles are in the zero energy states, half in each zero energy level.

High T. $\frac{n_1}{N} = \lim\limits_{T\to\infty} \frac{e^0}{2 + 2e^{-\frac{\varepsilon}{kT}} + e^{-\frac{2\varepsilon}{kT}}} = \frac{1}{5}$ $\quad \frac{n_2}{N} = \frac{1}{5}$

$$\frac{n_3}{N} = \lim_{T\to\infty} \frac{e^{-\frac{\varepsilon}{kT}}}{2 + 2e^{-\frac{\varepsilon}{kT}} + e^{-\frac{2\varepsilon}{kT}}} = \frac{1}{5} \qquad \frac{n_4}{N} = \frac{1}{5} \qquad \frac{n_5}{N} = \frac{1}{5} \qquad \overline{E} = \frac{4}{5}\varepsilon$$

Most disordered state: 20% in each level has the most ways to achieve this disorder.

**5. Stat Mechanics without a Calculator.** There are N = 26,000,000 particles, where some are in energy state $E_1 = 0$ and the rest are in state $E_2 = \varepsilon$. Give the occupation number $n_1$ for the 1[st] state when $\varepsilon = kT$, using the approximation $e = 2.718... \approx 2\frac{5}{7} = \frac{19}{7}$.

**Solution.** $E_1 = 0$, $E_2 = \varepsilon$, $\varepsilon = kT$, $N = 26{,}000{,}000$, $e \approx 19/7$.

$$Z = \sum_i e^{-\frac{E_i}{kT}} = 1 + e^{-\frac{\varepsilon}{kT}} = 1 + e^{-1} = 1 + \frac{7}{19} = \frac{26}{19}$$

$$n_1 = Ne^{-\frac{E_1}{kT}} / Z = N / Z = 26{,}000{,}000\left[\frac{19}{26}\right] = 19{,}000{,}000$$

6. **Entropy Change.** Write down how $\Delta S$ relates to $\Delta Q$. Now for an isochoric process, related $\Delta Q$ to $\Delta U$. For an ideal monatomic gas you can express $\Delta U$ in terms of $\Delta T$. Put these pieces together to show that the entropy change for an isochoric process where the temperature increases from $T_1$ to $T_2$ is given by $S_{1\to 2} = \frac{3}{2} nR \ln\frac{T_2}{T_1}$.

$$\Delta S = \Delta Q / T \qquad \Delta U = \Delta Q - P\Delta V$$

$$\Delta Q_V = \Delta U \quad \text{(constant volume, isochoric)}$$

$$\Delta S_V = \frac{\Delta Q_V}{T} = \Delta S_V = \frac{\Delta U}{T}$$

$$U = \frac{3}{2} nRT \qquad \Delta U = \frac{3}{2} nR\Delta T$$

$$\Delta S_V = \frac{\Delta U}{T} = \frac{3}{2} nR \frac{\Delta T}{T} \qquad dS_V = \frac{3}{2} nR \frac{dT}{T}$$

$$S_{1\to 2} = \frac{3}{2} nR \int_{T_1}^{T_2} \frac{dT}{T} = \frac{3}{2} nR \ln T \Big|_{T_1}^{T^2} = \frac{3}{2} nR \ln\frac{T_2}{T_1} \quad \text{where } \Delta V = 0$$

7. **Entropy and Combining Systems.** Write down the formula that relates the entropy $S$ to the number of ways $\Omega$ a state can be in. Introduce the new modified Boltzmann constant $k_{10}$ so that you can replace ln with log, i.e., base 10. What is the entropy in terms of $k_{10}$ for system A that has a billion microstates? What is the entropy in terms of $k_{10}$ for system B that has a trillion microstates? What is the entropy of the combined systems?

$$S = k \ln \Omega = k_{10} \log \Omega$$

$$S_A = k_{10} \log 10^9 = 9k_{10} \qquad S_B = k_{10} \log 10^{12} = 12k_{10}$$

$$S_{\text{combined}} = k_{10} \log(\Omega_A \Omega_B) = k_{10} \log \Omega_A + k_{10} \log \Omega_B = 21k_{10}$$

## I. Quantization.

**1. Particle in a One-Dimensional Box.** Write down the kinetic energy for a particle in terms of momentum p and mass m. There is no potential energy here. Use the de Broglie relation to replace p and introduce wavelength. Quantize the wavelength so n half-waves fit in length L. Derive the discrete energy spectrum

$$E_n = \frac{n^2 h^2}{8mL^2}$$

$$E = \frac{1}{2} m v^2 \qquad E = \frac{1}{2} \frac{m^2 v^2}{m} = \frac{p^2}{2m} \qquad p = \frac{h}{\lambda} \qquad E = \left[\frac{h}{\lambda}\right]^2 \frac{1}{2m}$$

$$n \frac{\lambda}{2} = L \qquad \lambda = \frac{2L}{n} \qquad \frac{1}{\lambda} = \frac{n}{2L}$$

$$E = \frac{p^2}{2m} = \frac{1}{2m} \frac{h^2}{\lambda^2} = \frac{1}{2m} h^2 \left[\frac{n}{2L}\right]^2 = \frac{n^2 h^2}{8mL^2}$$

$$E_n = \frac{n^2 h^2}{8mL^2} \quad \text{or} \quad E_n = \frac{n^2 h^2}{8mL^2} = \frac{n^2 (2\pi\hbar)^2}{8mL^2} = \frac{n^2 4\pi^2 \hbar^2}{8mL^2} = \frac{n^2 \pi^2 \hbar^2}{2mL^2}$$

**2. Feynman's Derivation of the Bohr Radius**. Write down the classical form of kinetic and potential energy in terms of the momentum p, particle mass m, and potential energy V. Take m to be an electron in a classical circular orbit around a proton. Classical electromagnetic theory says the accelerating electron will radiate and spiral into the proton. Like Bohr, we postulate that this destruction of the hydrogen atom will not happen. The potential energy for the Coulomb attraction between the electron and proton is $V(r) = -\frac{e^2}{4\pi\varepsilon_0 r}$, where $e$ is the absolute value of the charge of the electron (as well as for the proton), and $r$ is the distance between the proton and the electron. Since my chemistry teacher told me that the mass of the proton is 1836 times the mass of the electron, we can consider that the proton does not move. Replace p using the de Broglie relation. Fix (quantize) the wavelength by requiring that one wavelength be equal to the circumference of the orbit. Find the radius that minimizes the energy.

$$E = \frac{p^2}{2m} + V$$

$$E = \frac{p^2}{2m} - \frac{e^2}{4\pi\varepsilon_0 r}$$

$$\lambda = \frac{h}{p} \quad \text{and} \quad p = \frac{h}{\lambda}$$

$$\lambda = 2\pi r \quad \text{and} \quad p = \frac{h}{2\pi r} = \frac{\hbar}{r}$$

$$E = \frac{\hbar^2}{2mr^2} - \frac{e^2}{4\pi\varepsilon_0 r}$$

$$\frac{dE}{dr} = -\frac{\hbar^2}{mr^3} + \frac{e^2}{4\pi\varepsilon_0 r^2} = 0$$

$$\frac{\hbar^2}{mr^3} = \frac{e^2}{4\pi\varepsilon_0 r^2} \qquad r = \frac{4\pi\varepsilon_0 \hbar^2}{me^2} \equiv a$$

The Bohr Radius a =0.529 Å.

This calculation is a lucky coincidence since one needs quantum for the precise work.

## J. Spinors.

**1. Matrix Definitions.** Given $A = \begin{bmatrix} 1+i & 2+i \\ 2-i & 1-i \end{bmatrix}$, find $Tr(A)$, $A^T$, $A*$, $A^\dagger$, $|A|$.

$$Tr(A) = 1+i+1-i = 2$$

$$A^T = \begin{bmatrix} 1+i & 2-i \\ 2+i & 1-i \end{bmatrix}$$

$$A* = \begin{bmatrix} 1-i & 2-i \\ 2+i & 1+i \end{bmatrix}$$

$$A^\dagger = \begin{bmatrix} 1-i & 2-i \\ 2+i & 1+i \end{bmatrix}$$

$$|A| = (1+i)(1-i)-(2-i)(2+i) = (1-i^2)-(4-i^2) = 2-5 = -3$$

**2. Commutators.** Given $A = \begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}$ and $B = \begin{bmatrix} i & 1 \\ 1 & -i \end{bmatrix}$, find $[A, B]$, $\{A, B\}$.

$$AB = \begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}\begin{bmatrix} i & 1 \\ 1 & -i \end{bmatrix} = \begin{bmatrix} i+i & 1-i^2 \\ -i^2+1 & -i-i \end{bmatrix} = \begin{bmatrix} 2i & 2 \\ 2 & -2i \end{bmatrix}$$

$$BA = \begin{bmatrix} i & 1 \\ 1 & -i \end{bmatrix}\begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix} = \begin{bmatrix} i-i & i^2-1 \\ 1+i^2 & i-i \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix}$$

$$[A, B] = \begin{bmatrix} 2i & 4 \\ 2 & -2i \end{bmatrix} \qquad \{A, B\} = \begin{bmatrix} 2i & 0 \\ 2 & -2i \end{bmatrix}$$

**3. Eigenvectors and Eigenvalues.** Find the normalized eigenvectors and associated eigenvalues for $\sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$ .

$$\det\begin{bmatrix} -\lambda & -i \\ i & -\lambda \end{bmatrix} = 0$$

$$\lambda^2 - 1 = 0$$

$$\lambda = \pm 1$$

$$\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \lambda \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

For $\lambda = +1$, we get $-ic_2 = c_1$ for the upper component of the matrix multiplication.

Pick $c_1 = 1$ and normalize later. Then $c_2 = ic_1 = i$ and $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ i \end{bmatrix}$. For normalization: $c_1^* c_1 + c_2^* c_2 = 1 \cdot 1 + (-i)(i) = 2$ to be 1 instead. Therefore, $u = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ i \end{bmatrix}$ is normalized. For $\lambda = -1$, we get $-ic_2 = -c_1$. Then $c_2 = -ic_1$ and $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -i \end{bmatrix}$.

Normalization leads to $v = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -i \end{bmatrix}$,

Summary: the normalized eigenvectors or eigenspinors are

Eigenvalue +1 case: $u = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ i \end{bmatrix}$ Eigenvalue -1 case: $v = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -i \end{bmatrix}$

## K. The Pauli Equation.

### 1. Eigenfunctions and Eigenvalues.

Consider the Hamiltonian operator $H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V$ where $V = 0$ and the function $\psi_3(x) = A\sin(3\pi x / L)$. Show that $\psi_3(x)$ is an eigenfunction of $H$ and find the eigenvalue.

$$H\psi_3 = -\frac{\hbar^2}{2m}\frac{d^2\psi_3}{dx^2} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}[A\sin(3\pi x / L)]$$

$$H\psi_3 = -\frac{\hbar^2}{2m}\frac{d}{dx}[\frac{3\pi}{L}A\cos(3\pi x / L)]$$

$$H\psi_3 = \frac{\hbar^2}{2m}\left[\frac{3\pi}{L}\right]^2 A\sin(3\pi x / L)$$

$$H\psi_3 = \frac{9\pi^2\hbar^2}{2mL^2}\psi_3$$

We got the same thing back which means $\psi_3(x)$ is an eigenfunction of $H$.

The eigenvalue is what's in front: $\frac{9\pi^2\hbar^2}{2mL^2}$.

Do you recognize this eigenvalue as the third energy from the discrete energy spectrum?

$$E_n = \frac{n^2\pi^2\hbar^2}{2mL^2}$$

The eigenvalues of the Hamiltonian are energies.

**2. Commutators with Differential Operators.** In class we showed that $[x,\, p]\psi = i\hbar\psi$, i.e., $[x,\, p] = i\hbar$. Express the commutator $\left[x^2, x\dfrac{d}{dx}\right]$ in simplest terms.

$$\left[x^2, x\frac{d}{dx}\right]\psi = x^2(x\frac{d\psi}{dx}) - x\frac{d}{dx}(x^2\psi) = x^3\frac{d\psi}{dx} - x(2x\psi) - x^3\frac{d\psi}{dx}$$

$\left[x^2, x\dfrac{d}{dx}\right]\psi = -x(2x\psi)$. Therefore, $\left[x^2, x\dfrac{d}{dx}\right] = -2x^2$

## L. The Dirac Equation.

**1. Dirac Four-Spinor.** When a spin-1/2 is moving, the Dirac four spinor for the particle is given by

$$u(\vec{p}, s) = \sqrt{\frac{E+mc^2}{2E}}\begin{bmatrix} \phi(s) \\ \dfrac{c\,\vec{\sigma}\cdot\vec{p}}{E+mc^2}\phi(s) \end{bmatrix}$$, where $\phi(s) = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$.

Express this four-spinor in simplest terms for the case where

$E = 2$, $mc^2 = 1$, $c\vec{p} = \hat{i} + \hat{j} + \hat{k}$, and $\phi(s) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

$$u(\vec{p}, s) = \sqrt{\frac{E+mc^2}{2E}}\begin{bmatrix} \phi(s) \\ \dfrac{c\,\vec{\sigma}\cdot\vec{p}}{E+mc^2}\phi(s) \end{bmatrix} \qquad \sqrt{\frac{E+mc^2}{2E}} = \sqrt{\frac{2+1}{2(2)}} = \frac{\sqrt{3}}{2}$$

$$c\,\vec{\sigma}\cdot\vec{p} = \sigma_x + \sigma_y + \sigma_z = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1-i \\ 1+i & -1 \end{bmatrix}$$

$$\frac{c\,\vec{\sigma}\cdot\vec{p}}{E+mc^2}\phi(s) = \frac{1}{3}\begin{bmatrix} 1 & 1-i \\ 1+i & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 \\ 1+i \end{bmatrix}$$ and $$u(\vec{p}, s) = \frac{\sqrt{3}}{2}\begin{bmatrix} 1 \\ 0 \\ 1/3 \\ (1+i)/3 \end{bmatrix}$$

## M. The Method of Frobenius.

**1. Recurrence Relation.** For the series $y(x) = \sum_{k=0}^{\infty} a_k x^k$ , the recurrence relation $a_{k+2} = -\frac{1}{(k+2)(k+1)} a_k$ is found. Find the solution for the odd power series where $a_0 = 0$ and $a_1 = 1$. Show the first four terms explicitly.

$$a_{k+2} = -\frac{1}{(k+2)(k+1)} a_k \qquad a_3 = -\frac{1}{(1+2)(1+1)} a_1 = -\frac{1}{3\cdot 2}$$

$$a_5 = -\frac{1}{(3+2)(3+1)} a_3 = -\left[\frac{1}{5\cdot 4}\right] a_3 = -\left[\frac{1}{5\cdot 4}\right]\left[-\frac{1}{3\cdot 2}\right] = \frac{1}{5!}$$

$$a_7 = -\frac{1}{(5+2)(5+1)} a_5 = -\left[\frac{1}{7\cdot 6}\right] a_5 = -\left[\frac{1}{7\cdot 6}\right]\frac{1}{5!} = -\frac{1}{7!}$$

The odd series solution is $g(x) = a_1 + a_3 x^3 + a_5 x^5 + a_7 x^7 + \ldots$

$$g(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots = \sin x$$

**2. Polynomials.** For the series $y(x) = \sum_{k=0}^{\infty} a_k x^k$ , the recurrence relation $a_{k+1} = \frac{(k-n)}{(k+1)^2} a_k$ is found. For the specific case where $a_0 = 1$ and $n = 1$, find the relevant polynomial y(x).

$$a_{k+1} = \frac{(k-1)}{(k+1)^2} a_k \qquad a_1 = a_{0+1} = \frac{(0-1)}{(0+1)^2} a_0 = -a_0$$

$$a_2 = a_{1+1} = \frac{(1-1)}{(1+1)^2} a_1 = 0 \qquad y(x) = a_0 + a_1 x \qquad y(x) = 1 - x$$

**N. The Dirac Delta Function.**

**1. Applying the Dirac Delta Function** $\delta(x)$ **.** Evaluate $\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}} \delta(x)\, dx$ .

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}} \delta(x)\, dx = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}} \Bigg|_{x=0} = \frac{1}{\sqrt{2\pi\sigma^2}}$$

**2. Applying Dirac Delta Function** $\delta(x-a)$ **.** Find $\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}} \delta(x-a)\, dx$ .

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}} \delta(x-a)\, dx = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}} \Bigg|_{x=a} = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{a^2}{2\sigma^2}}$$

**3. A Dirac Delta Identity.** Evaluate the following two integrals, showing all steps.

$$I_{k>0} = \int_{-\infty}^{\infty} f(x)\,\delta(kx)\, dx \text{ , where } k > 0$$

$$I_{k<0} = \int_{-\infty}^{\infty} f(x)\,\delta(kx)\, dx \text{ , where } k < 0$$

Let z = kx and use what you know about the delta function from class.

$$I_{k>0} = \int_{-\infty}^{\infty} f(\frac{z}{k})\,\delta(z)\frac{dz}{k} = \frac{1}{k}\int_{-\infty}^{\infty} g(z)\,\delta(z)\, dz = \frac{1}{k} g(0) = \frac{f(0)}{k}$$

$$I_{k<0} = \int_{\infty}^{-\infty} f(\frac{z}{k})\,\delta(z)\frac{dz}{k} = -\int_{-\infty}^{\infty} f(\frac{z}{k})\,\delta(z)\frac{dz}{k}$$ switching integration limits.

$$I_{k<0} == -\frac{1}{k}\int_{-\infty}^{\infty} g(z)\,\delta(z)\, dz = -\frac{1}{k} g(0) = \frac{f(0)}{-k}$$

Both cases can be combined as $\int_{-\infty}^{\infty} f(x)\,\delta(kx) dx = \frac{f(0)}{|k|}$