Theoretical Physics **Exam 3** **April 11, 2020**
Prof. Ruiz, UNC Asheville **Study Guide**

Overview/Goals for the Exam: Memorization & Calculation Skills.

Class O. Fourier Series. Fourier's Theorem is first introduced in a conceptual way. Then, the mathematical procedure is developed. The Fourier series for the periodic square wave is calculated. After completing this module you should be able to do the following.

1. Write down from memory the general form for a Fourier series and the amplitude formulas.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos(nx) + b_n \sin(nx) \right]$$

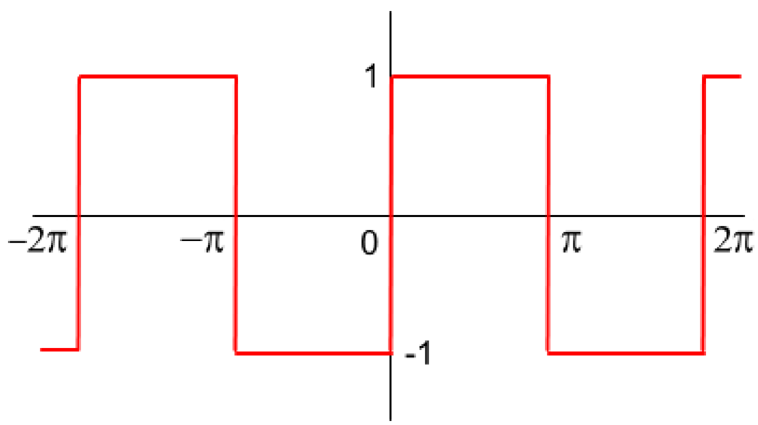
$$a_0 = \frac{1}{\pi} \int_{-\pi}^{+\pi} f(x)\, dx \qquad a_n = \frac{1}{\pi} \int_{-\pi}^{+\pi} f(x) \cos(nx)\, dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{+\pi} f(x) \sin(nx)\, dx$$

2. Calculate Fourier amplitudes for periodic functions over the interval $-\pi \le x \le \pi$.

Sample Questions with Fourier Series

Fourier Series 1 - Square Wave. Find the Fourier Series for the periodic square wave shown below.



For full credit, write out your answer by giving the first five nonzero terms. You must write f(x) = and then give the coefficients multiplied by the appropriate trig function for 5 nonzero terms where each coefficient is in simplest mathematical form.

Solution. Since the above square wave is an odd function, the a_0 and a_n integrals are zero. The b_n integral is the one that will give nonzero values.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos(nx) + b_n \sin(nx)\right]$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\sin(nx)dx = \frac{2}{\pi}\int_0^{\pi}\sin(nx)dx$$

$$b_n = -\frac{2}{\pi}\frac{\cos(nx)}{n}\Big|_0^{\pi} = -\frac{2}{\pi}\frac{1}{n}\left[\cos(n\pi) - \cos(0)\right]$$

For even $n = 2k$, where $k = 1, 2, 3...$

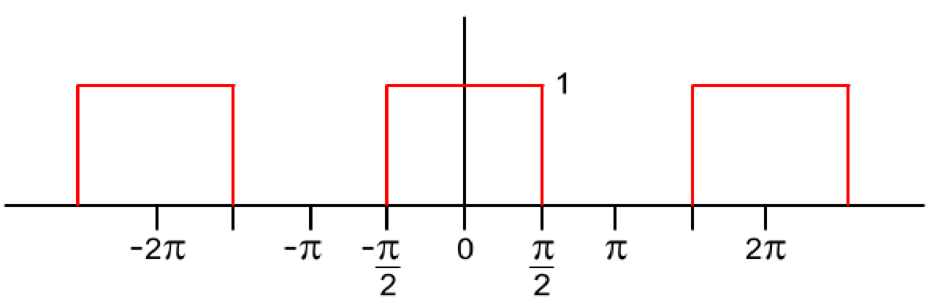
$$b_{2k} = -\frac{2}{\pi}\frac{1}{2k}\left[\cos(2\pi k) - \cos(0)\right] = -\frac{2}{\pi}\frac{1}{2k}(1-1) = 0$$

For even $n = 2k-1$, where $k = 1, 2, 3...$ $b_{2k-1} = -\frac{2}{\pi}\frac{1}{2k-1}\left[\cos(2\pi k - \pi) - \cos(0)\right]$

$$= -\frac{2}{\pi}\frac{1}{2k-1}(-1-1) = \frac{4}{\pi}\frac{1}{2k-1} \qquad b_n = \frac{4}{\pi}\frac{1}{n} \text{ for odd } n.$$

$$f(x) = \frac{4}{\pi}\left[\sin x + \frac{1}{3}\sin(3x) + \frac{1}{5}\sin(5x) + \frac{1}{7}\sin(7x)...\right]$$

Fourier Series 2 – Train of Pulses. Find the Fourier Series for the periodic wave shown below.



For full credit, write out your answer by giving the first five nonzero terms. You must write f(x) = and then give the coefficients multiplied by the appropriate trig function for 5 nonzero terms where each coefficient is in simplest mathematical form.

Solution. Since the above square wave is an even function, the b_n integral gives zero. The a_0 and a_n integrals are nonzero.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos(nx) + b_n \sin(nx)\right]$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\,dx \qquad a_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\cos(nx)dx \quad b_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\sin(nx)dx$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} 1\,dx = \frac{2}{\pi}\int_{0}^{\pi/2} dx = \frac{2}{\pi}\,x\Big|_0^{\pi/2} = \frac{2}{\pi}\frac{\pi}{2} = 1$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\cos(nx)dx = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} 1\cdot\cos(nx)dx = \frac{2}{\pi}\int_{0}^{\pi/2}\cos(nx)dx$$

$$a_n = \frac{2}{\pi}\frac{1}{n}\sin(nx)\Big|_0^{\pi/2} = \frac{2}{\pi}\frac{1}{n}\left[\sin(\frac{n\pi}{2}) - 0\right] = \frac{2}{\pi}\frac{1}{n}\sin(\frac{n\pi}{2})$$

$$a_1 = \frac{2}{\pi}\sin(\frac{\pi}{2}) = \frac{2}{\pi} \qquad a_2 = \frac{2}{\pi}\frac{1}{2}\sin(\frac{2\pi}{2}) = \frac{1}{2}\frac{2}{\pi}\sin(\pi) = 0$$

$$a_3 = \frac{2}{\pi}\frac{1}{3}\sin(\frac{3\pi}{2}) = -\frac{1}{3}\frac{2}{\pi} \qquad a_4 = \frac{2}{\pi}\frac{1}{4}\sin(\frac{4\pi}{2}) = \frac{2}{\pi}\frac{1}{4}\sin(2\pi) = 0$$

$$a_5 = \frac{2}{\pi}\frac{1}{5}\sin(\frac{5\pi}{2}) = \frac{2}{\pi}\frac{1}{5} \qquad a_6 = \frac{2}{\pi}\frac{1}{6}\sin(\frac{6\pi}{2}) = \frac{2}{\pi}\frac{1}{6}\sin(3\pi) = 0$$

$$a_7 = \frac{2}{\pi}\frac{1}{7}\sin(\frac{7\pi}{2}) = \frac{2}{\pi}\frac{1}{7}\sin(\frac{3\pi}{2}) = -\frac{2}{\pi}\frac{1}{7}$$

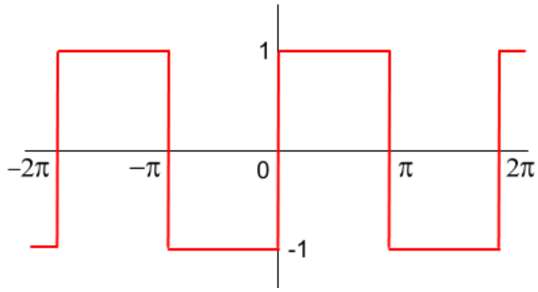
Remember $a_0 = 1$ => $\frac{a_0}{2} = \frac{1}{2}$.

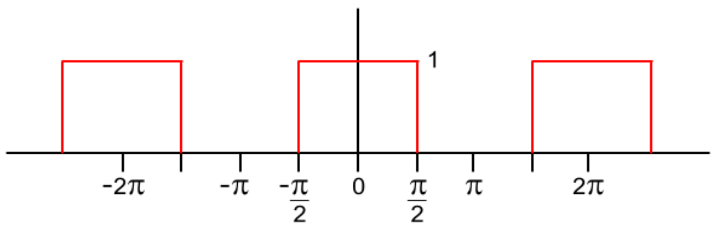
$$f(x) = \frac{a_0}{2} + a_1\cos x + a_2\cos(2x) + a_3\cos(3x) + a_4\cos(4x)\ldots$$

$$f(x) = \frac{1}{2} + \frac{2}{\pi}\left[\cos x - \frac{1}{3}\cos(3x) + \frac{1}{5}\cos(5x) - \frac{1}{7}\cos(7x)\ldots\right]$$

OBSERVATION

Say I were publishing a paper on this result or had to derive the result for a job. I need to know if it is correct. And say the regular square wave case was already published or that I could look it up.





$$f(x)=\frac{4}{\pi}\left[\sin x+\frac{1}{3}\sin(3x)+\frac{1}{5}\sin(5x)+\ldots\right]$$

I need to transform the regular square wave case into the one that I need.

So I raise the first wave by 1 and multiply by 1/2 to shrink it vertically down to size.

$$g(x)=\frac{1}{2}+\frac{2}{\pi}\left[\sin x+\frac{1}{3}\sin(3x)+\frac{1}{5}\sin(5x)+\ldots\right].$$

Then I shift to the left by $\frac{\pi}{2}$ using $x+\frac{\pi}{2}$.

$$h(x)=f(x+\frac{\pi}{2})=\frac{1}{2}+\frac{2}{\pi}\left[\sin(x+\frac{\pi}{2})+\frac{1}{3}\sin(3x+\frac{3\pi}{2})+\frac{1}{5}\sin(5x+5\frac{\pi}{2})+\ldots\right]$$

Complementary angles => $\sin x=\cos(\frac{\pi}{2}-x)$ so $\sin x=\cos(x-\frac{\pi}{2})$ since cos is even.

Does this identity work for all angles x? Think about that as we proceed.

$$\sin(x+\frac{\pi}{2})=\cos(x+\frac{\pi}{2}-\frac{\pi}{2})=\cos x$$

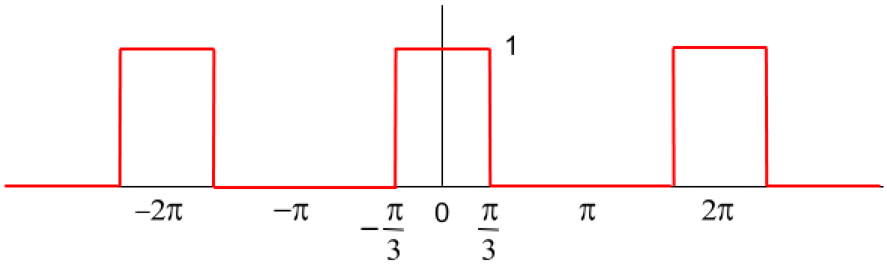
$$\sin(3x+\frac{3\pi}{2})=\cos(3x+\frac{3\pi}{2}-\frac{\pi}{2})=\cos(3x+\pi)=-\cos(3x)$$

$$\sin(5x+\frac{5\pi}{2})=\cos(5x+\frac{5\pi}{2}-\frac{\pi}{2})=\cos(5x+2\pi)=\cos(5x)$$

$$\sin(7x+\frac{7\pi}{2})=\cos(7x+\frac{7\pi}{2}-\frac{\pi}{2})=\cos(7x+3\pi)=-\cos(7x)$$

$$h(x)=\frac{1}{2}+\frac{2}{\pi}\left[\cos x-\frac{1}{3}\cos(3x)+\frac{1}{5}\cos(5x)-\frac{1}{7}\cos(7x)\ldots\right]$$

Fourier Series 3 – Pulse Train with 33.33…% Duty Cycle. Find the Fourier Series for the periodic wave shown below. Your basic cycle for this repeating pattern is defined over our standard region $-\pi \le x \le \pi$, symmetric about the origin, where the pulse is 1/3 the period (or wavelength) of the periodic wave.



Pulse Train with Duty Cycle $33\frac{1}{3}\%$

For full credit, write out your answer by giving the first six nonzero terms. You must write f(x) = and then give the coefficients multiplied by the appropriate trig function for your nonzero terms where each coefficient is in simplest mathematical form.

Solution. Pulse Train with $33\frac{1}{3}\%$ duty cycle with pulse center on zero. This placement means an even function so that we only have the constant and the cosines.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos(nx) + b_n \sin(nx) \right]$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\,dx \qquad a_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\cos(nx)\,dx \quad b_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\sin(nx)\,dx$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\,dx = \frac{2}{\pi}\int_{0}^{\pi/3} 1\cdot dx = \frac{2}{\pi} x\Big|_0^{\pi/3} = \frac{2}{\pi}\frac{\pi}{3} = \frac{2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(x)\cos(nx)\,dx = \frac{2}{\pi}\int_{0}^{\pi/3} \cos(nx)\,dx = \frac{2}{\pi}\frac{\sin(nx)}{n}\Bigg|_0^{\pi/3} = \frac{2}{\pi}\frac{\sin(n\pi/3)}{n}$$

For n = 1, 2, 3, 4, 5, 6, 7, ... $\sin(n\pi/3)$ gives $\frac{\sqrt{3}}{2}$, $\frac{\sqrt{3}}{2}$, 0, $-\frac{\sqrt{3}}{2}$, $-\frac{\sqrt{3}}{2}$, 0, $\frac{\sqrt{3}}{2}$, ...

$$f(x) = \frac{1}{3} + \frac{\sqrt{3}}{\pi}\left[\cos x + \frac{\cos 2x}{2} - \frac{\cos 4x}{4} - \frac{\cos 5x}{5} + \frac{\cos 7x}{7} + \ldots \right]$$

Class P. Fourier Transforms. The concept of the Fourier transform is developed from Fourier series. After completing this module you should be able to do the following.

1. Write down from memory the Fourier transform and inverse Fourier transform formulas.

$$\Im\{f(x)\} \equiv F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) e^{-ikx} dx$$

$$\Im^{-1}\{F(k)\} \equiv f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(k) e^{ikx} dk$$

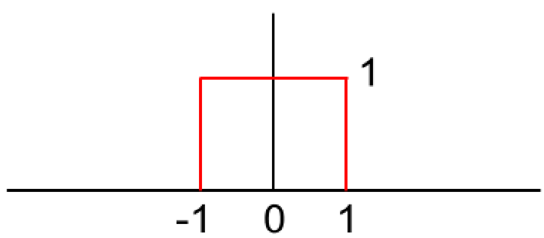
2. Calculate Fourier transforms and inverse Fourier transforms.

Sample Questions with Fourier Transforms

Fourier Transform 1 – Fourier Transform of the Delta Function. Find the Fourier transform of $\delta(x)$.

Solution. $$\Im\{f(x)\} \equiv F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \delta(x) e^{-ikx} dx = \frac{1}{\sqrt{2\pi}} e^{-ikx}\Big|_{x=0} = \frac{1}{\sqrt{2\pi}}$$

Fourier Transform 2 – Fourier Transform of Box Function. Find the Fourier transform of the box function below.



Solution. $$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) e^{-ikx} dx = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{-ikx} dx$$

$$F(k) = \frac{1}{\sqrt{2\pi}} \frac{e^{-ikx}}{(-ik)}\Bigg|_{-1}^{1} = \frac{1}{\sqrt{2\pi}} \left[\frac{e^{-ikx} - e^{ikx}}{-ik}\right] = \frac{1}{\sqrt{2\pi}} \left[\frac{e^{ik} - e^{-ik}}{ik}\right]$$

$$F(k) = \frac{1}{\sqrt{2\pi}} \frac{2}{k} \left[\frac{e^{ik} - e^{-ik}}{2i}\right] = \sqrt{\frac{2}{\pi}} \frac{1}{k} [\sin(k)] = \sqrt{\frac{2}{\pi}} \mathrm{sinc}(k)$$

Fourier Transform 3 – Fourier Transform of an Exponential Function. Find the Fourier Transform for the following function.

$$f(x) = e^{-ax} \text{ for } x \geq 0 \text{ where } a > 0 \text{ and } f(x) = 0 \text{ for } x < 0 .$$

Solution.

$$\Im\{f(x)\} \equiv F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) e^{-ikx} dx$$

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{0}^{+\infty} e^{-ax} e^{-ikx} dx$$

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{0}^{+\infty} e^{-ax-ikx} dx$$

$$F(k) = \frac{1}{\sqrt{2\pi}} \left[\frac{1}{-\alpha - ik} \right] e^{-ax-ikx} \Big|_{0}^{\infty}$$

$$F(k) = \frac{1}{\sqrt{2\pi}} \left[\frac{1}{-\alpha - ik} \right] (0-1)$$

$$F(k) = \frac{1}{\sqrt{2\pi}} \left[\frac{1}{\alpha + ik} \right]$$ or, you can continue with

$$F(k) = \frac{1}{\sqrt{2\pi}} \left[\frac{1}{\alpha + ik} \right] \left[\frac{\alpha - ik}{\alpha - ik} \right] = \frac{1}{\sqrt{2\pi}} \frac{\alpha - ik}{\alpha^2 + k^2}$$

$$F(k) = \frac{1}{\sqrt{2\pi}} \frac{\alpha - ik}{\alpha^2 + k^2} .$$

Fourier Transform 4 – Fourier Transform of an Exponential Function with Absolute Value. Find the Fourier transform of $f(x) = e^{-a|x|}$ where $a > 0$.

Solution.

$$\Im\{f(x)\} \equiv F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) e^{-ikx} dx$$

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-a|x|} e^{-ikx} dx$$

$$F(k) = \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{0} e^{+ax} e^{-ikx} dx + \int_{0}^{\infty} e^{-ax} e^{-ikx} dx \right]$$

$$F(k) = \frac{1}{\sqrt{2\pi}} \left[\frac{1}{\alpha - ik} \right] e^{ax-ikx} \Big|_{-\infty}^{0} + \frac{1}{\sqrt{2\pi}} \left[\frac{1}{-\alpha - ik} \right] e^{-ax-ikx} \Big|_{0}^{\infty}$$

$$F(k) = \frac{1}{\sqrt{2\pi}} \left[\frac{1}{\alpha - ik} \right] (1-0) + \frac{1}{\sqrt{2\pi}} \left[\frac{1}{-\alpha - ik} \right] (0-1)$$

$$F(k) = \frac{1}{\sqrt{2\pi}} \left[\frac{1}{\alpha - ik} \right] + \frac{1}{\sqrt{2\pi}} \left[\frac{1}{\alpha + ik} \right]$$

$$F(k) = \frac{1}{\sqrt{2\pi}} \left[\frac{1}{\alpha - ik} + \frac{1}{\alpha + ik} \right]$$

$$F(k) = \frac{1}{\sqrt{2\pi}} \left[\frac{\alpha + ik + \alpha - ik}{(\alpha - ik)(\alpha + ik)} \right]$$

$$F(k) = \frac{1}{\sqrt{2\pi}} \frac{2\alpha}{(\alpha^2 + k^2)}$$

Class Q. Laplace Transforms. The concept of the Laplace transform is developed. We take some Laplace transforms to construct a table. Then we use Laplace transforms to solve a differential equation. After completing this module you should be able to do the following.

1. Write down from memory the general form for a Laplace transform.

$$L\{f(t)\} = F(s) = \int_0^\infty f(t)e^{-st}dt \text{ , where } s > 0 \text{ .}$$

2. Calculate Laplace transforms.

3. Write down from memory the shifting property for Laplace transforms:

$$\text{If } g(t) = e^{at} f(t) \text{, then } G(s) = F(s-a) \text{, where } s > a \text{ .}$$

4. Write down from memory the Laplace transform for derivatives.

$$L\{f'(t)\} = sF(s) - f(0)$$

$$L\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$$

5. Solve differential equations using Laplace transforms.

Sample Questions with Laplace Transforms

Laplace Transform 1 – The Laplace Transform of 1. Find the Laplace transform of 1.

Solution.

Laplace function of interest is $f(t) = 1$.

$$F(s) = \int_0^\infty f(t)e^{-st}dt = \int_0^\infty e^{-st}dt$$

$$F(s) = \left.\frac{e^{-st}}{-s}\right|_0^\infty = 0 - \left[\frac{1}{-s}\right]$$

$$F(s) = \frac{1}{s} \text{ , where } s > 0 \text{ .}$$

Laplace Transform 2 – The Laplace Transform of an Exponential. Find the Laplace transform of $f(t) = e^{at}$.

Solution.

$$F(s) = \int_0^\infty e^{at} e^{-st} dt = \int_0^\infty e^{(a-s)t} dt$$. We must have $s > a$.

$$F(s) = \left. \frac{e^{(a-s)t}}{a-s} \right|_0^\infty = 0 - \left[\frac{1}{a-s} \right] = \frac{1}{s-a} \quad \text{for } s > a$$

Laplace Transform 3 – The Laplace Transform of Trig Functions. Find the Laplace transform for $f(t) = \cos \omega t$ and $g(t) = \sin \omega t$.

Solution. Let $h(t) = e^{i\omega t}$ and first find $H(s) = L\{e^{i\omega t}\}$.

$$H(s) = \int_0^\infty h(t) e^{-st} dt = \int_0^\infty e^{i\omega t} e^{-st} dt = \int_0^\infty e^{(i\omega - s)t} dt$$

$$H(s) = \left. \frac{e^{(i\omega - s)t}}{i\omega - s} \right|_0^\infty = 0 - \left[\frac{1}{i\omega - s} \right] = \frac{1}{s - i\omega}$$

$$H(s) = \frac{1}{s - i\omega} \left[\frac{s + i\omega}{s + i\omega} \right] = \frac{s + i\omega}{s^2 + \omega^2}$$

$$H(s) = F(s) + iG(s)$$

$$F(s) = L\{\cos(\omega t)\} = \frac{s}{s^2 + \omega^2}$$

$$G(s) = L\{\sin(\omega t)\} = \frac{\omega}{s^2 + \omega^2}$$

Laplace Transform 4 – The Laplace Transform of Powers of t. Find the Laplace transform for $f(t) = t^n$.

Solution.

$$F(s) = \int_0^\infty t^n e^{-st} dt$$

$$F(s) = \left[-\frac{d}{ds} \right]^n \int_0^\infty e^{-st} dt$$

$$F(s) = \left[-\frac{d}{ds} \right]^n L\{1\}$$, where $L\{1\} = \left. \frac{e^{-st}}{-s} \right|_0^\infty = 0 - \left[\frac{1}{-s} \right] = \frac{1}{s}$.

$$F(s) = \left[-\frac{d}{ds} \right]^n \frac{1}{s}$$

$$F(s) = \frac{n!}{s^{n+1}}$$

Laplace Transform 5 – The Laplace Transform Involving Derivatives. Given $f(t) = t^n$ and that $F(s) = \frac{n!}{s^{n+1}}$, use the Laplace transform of the derivative rule to find $L\{f'(t)\}$.

Solution. The Laplace transform derivative rule is $L\{f'(t)\} = sF(s) - f(0)$. Therefore,

$$L\{f'(t)\} = s\frac{n!}{s^{n+1}} - t^n \Big|_{t=0} = \frac{n!}{s^n}.$$

Let's check directly: $L\{f'(t)\} = L\{nt^{n-1}\} = nL\{t^{n-1}\} = n\frac{(n-1)!}{s^n} = \frac{n!}{s^n}$

Class R. Convolution. The concept of the convolution is developed using a radioactive decay scenario. We show that the convolution operation is commutative. We also show that the product of two Laplace transforms is equal to the Laplace transform of the convolution of the two original functions. This result can be summarized as

$$L[f(t)]\cdot L[g(t)] = L[f(t)*g(t)]$$

Finally, we present an analogy with power series. After completing this module you should be able to do the following.

1. Write down from memory the following formula for the convolution.

$$f(t)*g(t) = \int_0^t f(u)g(t-u)du$$

2. Evaluate convolutions of functions.

3. Write down from memory the following result for the Laplace transform of a convolution.

$$L\{f(t)*g(t)\} = F(s)G(s)$$

4. Take the Laplace transform of a convolution.

Sample Questions with Convolutions

Convolution 1 – The Convolution of 1 and a Function. Find the convolution $f(t)*g(t)$ where $f(t)=1$ and $g(t)=t$.

Solution.

$$f(t)*g(t) = \int_0^t f(u)g(t-u)du$$

$$f(t)*g(t) = \int_0^t 1\cdot(t-u)du = \int_0^t t\,du - \int_0^t u\,du$$

$$f(t)*g(t) = t\int_0^t du - \int_0^t u\,du$$

$$f(t)*g(t) = t\,u\Big|_0^t - \frac{u^2}{2}\Big|_0^t = t^2 - \frac{t^2}{2} = \frac{t^2}{2}$$

Convolution 2 – The Convolution of Two Powers. Find the convolution $f(t)*g(t)$ where $f(t)=t$ and $g(t)=t^2$.

Solution. $$f(t)*g(t)=\int_0^t f(u)g(t-u)du=\int_0^t u\cdot(t-u)^2 du$$

Since convolution commutes, I'd rather do

$$f(t)*g(t)=g(t)*f(t)=\int_0^t u^2\cdot(t-u)du=t\int_0^t u^2 du-\int_0^t u^3 du$$

$$f(t)*g(t)=t\left.\frac{u^3}{3}\right|_0^t-\left.\frac{u^4}{4}\right|_0^t$$

$$f(t)*g(t)=\frac{t^4}{3}-\frac{t^4}{4}=t^4\left[\frac{1}{3}-\frac{1}{4}\right]=t^4\left[\frac{4-3}{12}\right]$$

$$f(t)*g(t)=\frac{t^4}{12}$$

Convolution 3 – The Convolution with Delta Function. Show by explicit calculation that the convolutions $f(t)*g(t)$ and $g(t)*f(t)$ are equal, where the functions are $f(t)=f(t)$ and $g(t)=\delta(t)$. You will need the identity $\delta(t-u)=\delta(u-t)$.

Solution. We know that he delta function behaves as follows. Think sifting spike and shifted spike.

$$\int f(x)\delta(x)dx=f(0) \text{ and } \int f(x)\delta(x-a)dx=f(a)$$

Therefore, we have the following.

$$f(t)*g(t)=\int_0^t f(u)g(t-u)du=\int_0^t f(u)\cdot\delta(t-u)du$$

$$f(t)*g(t)=\int_0^t f(u)\cdot\delta(u-t)du=\left.f(u)\right|_{u=t}=f(t)$$

$$g(t)*f(t)=\int_0^t \delta(u)f(t-u)du=\left.f(t-u)\right|_{u=0}=f(t)$$

Convolution 4 – The Convolution and Laplace Transform. Use the result that $L\{t^n\} = \dfrac{n!}{s^{n+1}}$ to show that $L\{g(t) * f(t)\} = L\{g(t)\}L\{f(t)\} = G(s)F(s)$, where $f(t) = t^2$ and $g(t) = t^3$.

Solution. Note that calculating $L\{t^n\}$ from scratch was a previous problem in Section Q.

Here we use $L\{t^n\} = \dfrac{n!}{s^{n+1}}$ to obtain $F(s) = \dfrac{2!}{s^3} = \dfrac{2}{s^3}$ and $G(s) = \dfrac{3!}{s^4} = \dfrac{6}{s^4}$.

Then we calculate the convolution.

$$f(t) * g(t) = \int_0^t f(u)g(t-u)du$$

$$f(t) * g(t) = g(t) * f(t) = \int_0^t g(u) f(t-u)du$$

$$f(t) * g(t) = \int_0^t u^3 (t-u)^2 du$$

$$f(t) * g(t) = \int_0^t u^3 (t^2 - 2tu + u^2) du = \int_0^t (u^3 t^2 - 2tu^4 + u^5) du$$

$$f(t) * g(t) = \left[\frac{u^4}{4} t^2 - 2t \frac{u^5}{5} + \frac{u^6}{6} \right]_0^t = t^6 \left[\frac{1}{4} - \frac{2}{5} + \frac{1}{6} \right]$$

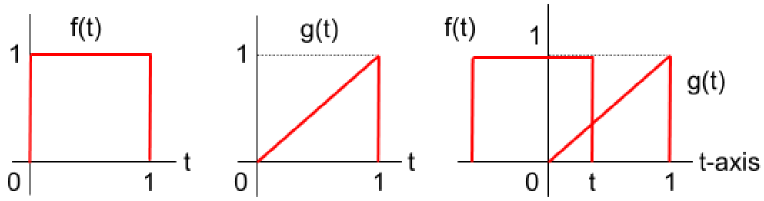
$$f(t) * g(t) = t^6 \left[\frac{15 - 24 + 10}{60} \right] = \frac{t^6}{60}$$

$$L\{f(t) * g(t)\} = L\{\frac{t^6}{60}\} = \frac{1}{60} L\{t^6\} = \frac{1}{60} \frac{6!}{s^7} = \frac{1}{60} \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2}{s^7} = \frac{12}{s^7}$$

$$F(s)G(s) = \frac{2}{s^3} \frac{6}{s^4} = \frac{12}{s^7}$$

Convolution 5 – Convolution Insight. Sketch the square pulse where $f(t)=1$ for $0 \le t \le 1$ and zero elsewhere. Sketch the ramp pulse where $g(t)=t$ for $0 \le t \le 1$ and zero elsewhere. Find the convolution $h(t)=f*g$ inside the nonzero region $0 \le t \le 1$. Note that you can do $h(t)=g*f$ if that is easier. To gain insight into the convolution, imagine the square pulse moving in from the far left and overlapping your ramp as the square pulse travels to the right. Sketch an overlap where the right edge of the square pulse is at some arbitrary t in the region $0 \le t \le 1$. Your left rectangular edge will be at $t-1$. What is the area of the common overlapped region?

SOLUTION.



$$f(t)*g(t)=\int_0^t f(u)g(t-u)du$$

$$f(t)*g(t)=\int_0^t 1\cdot(t-u)du$$

$$f(t)*g(t)=(tu-\frac{u^2}{2})\Big|_0^t$$

$$f(t)*g(t)=t^2-\frac{t^2}{2}=\frac{t^2}{2}$$

The overlap area above is $\frac{1}{2}(t)(t)=\frac{t^2}{2}=f*g$.

Class S. Cauchy Integral Formula. We introduce complex functions and define analytic functions with the Cauchy-Riemann conditions. We arrive at these conditions by requiring that differentiation and integration be unambiguous. We then derive the Cauchy Integral Formula. After completing this module you will be able to do the following.

1. Write down from memory the Cauchy-Riemann relations or conditions. $$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$$

2. Use the Cauchy-Riemann conditions to show that a given function is either analytic or not.

3. Write down from memory the Cauchy Integral Formula.

$$\oint \frac{f(z)}{z-a}dz=2\pi i f(a)$$

Sample Questions with Cauchy-Riemann Relations

Cauchy-Riemann 1 – Simplest Analytic Function. Show that $z = x + iy$ is analytic.

Solution. Need to show that the Cauchy-Riemann conditions are met.

$$u = x \quad \text{and} \quad v = y$$

$$\frac{\partial u}{\partial x} = 1 \qquad \frac{\partial v}{\partial y} = 1 \qquad \frac{\partial u}{\partial y} = 0 \qquad \frac{\partial v}{\partial x} = 0$$

Therefore: $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ and $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$.

The Cauchy-Riemann conditions are met. The function is analytic.

Cauchy-Riemann 2 – Quadratic Analytic Function. Show that $z^2 = (x + iy)^2$ is analytic.

Solution. Need to show that the Cauchy-Riemann conditions are met.

$$z^2 = (x + iy)^2 = x^2 + 2xyi - y^2$$

$$u = x^2 - y^2 \quad \text{and} \quad v = 2xy$$

$$\frac{\partial u}{\partial x} = 2x \qquad \frac{\partial v}{\partial y} = 2x \qquad \frac{\partial u}{\partial y} = -2y \qquad \frac{\partial v}{\partial x} = 2y$$

Therefore: $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ and $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$.

The Cauchy-Riemann conditions are met. The function is analytic.

Cauchy-Riemann 3 – Cauchy-Riemann Conditions Not Met. Give a function where the Cauchy-Riemann conditions are not met.

One Solution.

$$f(x, y) = x + xyi\ .$$

Class T. Poles and the Residue Theorem. We review the basics of complex functions, the Cauchy-Riemann relations, poles, and the Cauchy Integral Formula from the last chapter. Then we define residue and derive the residue theorem. Finally we apply residues to integration.

1. Write down from memory $\oint_C F(z)\,dz = 2\pi i \sum_n Res(F, z_n)$ and know how to use it, e.g.,

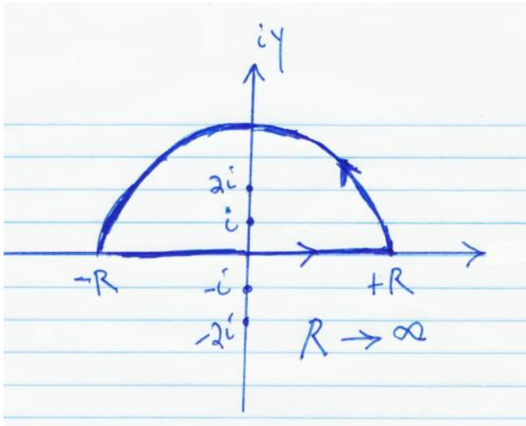
a)clear the n[th] singularity, b) set $z = z_n$, c) multiply by $2\pi i$.

2. Find poles for complex functions. 3. Calculate residues for complex functions (see 1 above).

4. Employ the residue theorem in evaluating integrals (see 1 above).

Sample Questions with Poles and the Residue Theorem

Poles 1 Complex Integration. Use complex integration to evaluate $I = \int_{-\infty}^{\infty} \frac{dx}{(x^2+1)(x^2+4)}$.



Solution. $F(z) = \frac{1}{(z^2+1)(z^2+4)}$ There are 4 poles.

$$z_1 = i \quad z_2 = 2i \quad z_3 = -i \quad z_4 = -2i$$

$$F(z) = \frac{1}{(z-z_1)(z-z_2)(z-z_3)(z-z_4)}$$

$$F(z) = \frac{1}{(z-i)(z-2i)(z+i)(z+2i)}$$

Two poles enclosed: $I = 2\pi i \sum_n \mathrm{Res}(F, z_n) = 2\pi i\left[\mathrm{Res}(F, z_1) + \mathrm{Res}(F, z_2)\right]$

$$I = 2\pi i\left[\mathrm{Res}(F, i) + \mathrm{Res(F,2i)}\right]$$

$$\mathrm{Res}(F, i) = \left.\frac{1}{(z-2i)(z+i)(z+2i)}\right|_{z=i} = \left[\frac{1}{-i}\right]\left[\frac{1}{2i}\right]\left[\frac{1}{3i}\right] = \frac{1}{6i} = \frac{-i}{6}$$

$$\mathrm{Res}(F, 2i) = \left.\frac{1}{(z-i)(z+i)(z+2i)}\right|_{z=2i} = \left[\frac{1}{i}\right]\left[\frac{1}{2i}\right]\left[\frac{1}{3i}\right] = \frac{1}{-6i} = \frac{i}{12}$$

$$I = 2\pi i\left[\mathrm{Res}(F, i) + \mathrm{Res(F,2i)}\right] = 2\pi i\left[\frac{-i}{6} + \frac{i}{12}\right] = \pi\left[\frac{1}{3} - \frac{1}{6}\right] = \frac{\pi}{6}$$

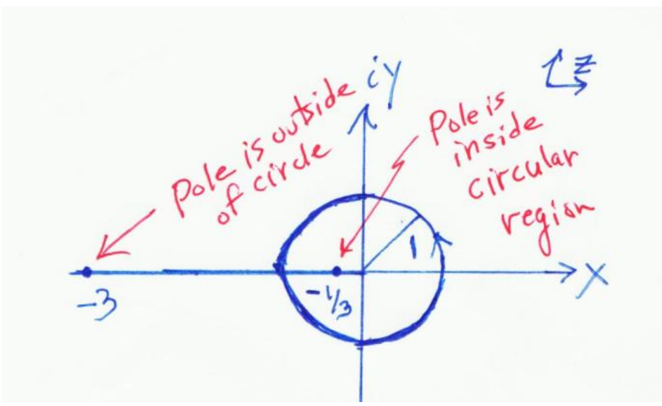
Poles 2 A Trig Angle Integration using Complex Integration. Evaluate $I = \int_0^{2\pi} \frac{d\theta}{5+3\cos\theta}$.

Solution. Let $z = e^{i\theta}$. Then $dz = ie^{i\theta}d\theta = izd\theta$ and $d\theta = dz/(iz)$

$$\cos\theta = \frac{1}{2}\left[e^{i\theta} + e^{-i\theta}\right] = \frac{1}{2}\left[z + \frac{1}{z}\right] \qquad I = \oint \frac{\frac{1}{iz}dz}{5 + \frac{3}{2}\left[z + \frac{1}{z}\right]}$$

$$I = \frac{1}{i}\oint \frac{dz}{5z + \frac{3}{2}\left[z^2 + 1\right]} \qquad I = \frac{2}{i}\oint \frac{dz}{3z^2 + 10z + 3}$$

Find Poles:



$$\frac{-10 \pm \sqrt{10^2 - 4(3)(3)}}{2(3)}$$

$$\frac{-10 \pm \sqrt{64}}{6} \text{ and } \frac{-10 \pm 8}{6}$$

$$-\frac{1}{3} \text{ and } -3$$

$$I = \frac{2}{i}\oint \frac{dz}{3z^2 + 10z + 3}$$. So let $F(z) = \frac{2}{i}\frac{1}{3(z + \frac{1}{3})(z+3)}$.

One pole inside: $I = 2\pi i \sum_n Res(F, z_n) = 2\pi i Res(F, -\frac{1}{3})$

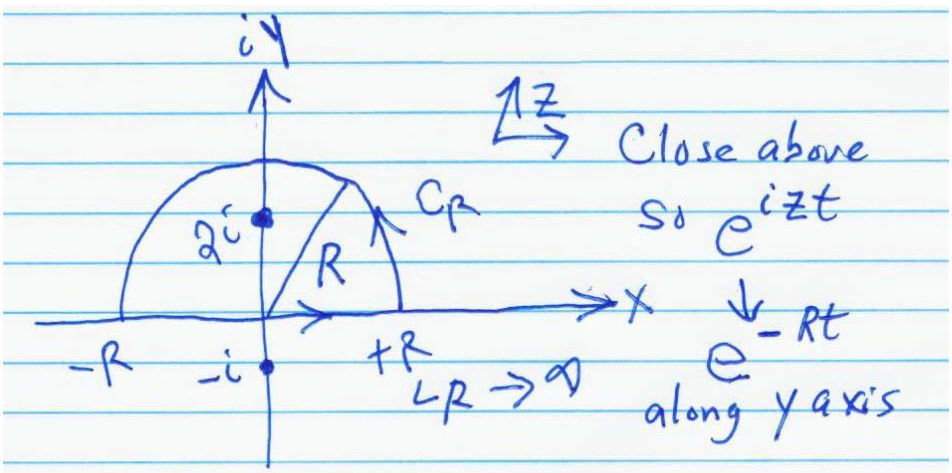
$$I = 2\pi i \frac{2}{i}\frac{1}{3(z+3)}\bigg|_{z=-\frac{1}{3}} = \frac{4\pi}{3}\frac{1}{[-\frac{1}{3}+3]} = \frac{4\pi}{3}\frac{1}{(8/3)} = \frac{\pi}{2}$$

Poles 3 Another Complex Integration. Evaluate $I(t) = \int_{-\infty}^{+\infty} \frac{e^{i\omega t}}{\omega^2 - i\omega + 2} d\omega$

Solution..

$$I(t) = \int_{-\infty}^{+\infty} \frac{e^{i\omega t}}{(\omega + i)(\omega - 2i)} d\omega$$

$$I(t) = \oint \frac{e^{izt}}{(z + i)(z - 2i)} dz$$



$$I(t) = 2\pi i Res\left[\frac{e^{izt}}{(z + i)(z - 2i)}, 2i\right]$$

$$I(t) = 2\pi i \left.\frac{e^{izt}}{(z + i)}\right|_{z=2i} = \frac{e^{-2t}}{3}$$