Class O. Fourier Series

O1. Fourier's Theorem

any waveform shape

One can construct $\underbrace{\text{any periodic wave}}$ having frequency f using sine waves with frequencies $f, 2f, 3f, 4f, \ldots$ (the harmonic series)

see qualitative figures in text or handout

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \sin(nx + \phi_n)$$

Remember our derivation $\sin(\alpha+\beta) = \cos\alpha \sin\beta + \sin\alpha \cos\beta$

$$\sin(nx+\phi_n) = \cos(nx)\sin\phi_n + \sin(nx)\cos\phi_n$$

$$f(x) = A_0 + \sum_{n=1}^{\infty}\Big[\underbrace{a_n}_{A_n \sin\phi_n}\cos(nx) + \underbrace{b_n}_{A_n \cos\phi_n}\sin(nx)\Big]$$

O2. Orthogonal Functions

$\int_{-\pi}^{\pi} \sin^2(nx)\,dx = \pi$ also

$$\int_{-\pi}^{\pi} \cos^2(nx)\,dx = \frac{1}{2}\int_{-\pi}^{\pi}\underbrace{\left[\cos^2(nx) + \sin^2(nx)\right]}dx = \frac{1}{2}x\Big|_{-\pi}^{\pi} = \pi$$

What about where $n \neq m$?

See Below 1

$$\int_{-\pi}^{\pi} \underbrace{\cos(nx)}_{\frac{1}{2}\left[e^{inx}+e^{-inx}\right]}\underbrace{\cos(mx)}_{\frac{1}{2}\left[e^{imx}+e^{-imx}\right]}\,dx = 0$$

Note: $e^{i\theta} = \cos\theta + i\sin\theta$
$e^{-i\theta} = \cos\theta - i\sin\theta$
$\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)$

Four integrals of the form

$$\int_{-\pi}^{\pi} e^{iPx}\,dx = \frac{1}{iP}e^{iPx}\Big|_{-\pi}^{\pi} = \frac{1}{iP}\left[\cos(Px) + i\sin(Px)\right]_{-\pi}^{\pi}$$

with $P \neq 0$

$\cos(P\pi) - \underbrace{\cos(-P\pi)}_{\cos(P\pi)} = 0$ even function

$\sin(P\pi) = 0$ always

Same goes for

Summarizes it all.

$$\int_{-\pi}^{\pi} \cos(nx)\sin(mx)\,dx = 0$$

Homework

$$\int_{-\pi}^{\pi} \cos(nx)\cos(mx)\,dx = \pi\delta_{nm}$$

$$\int_{-\pi}^{\pi} \sin(nx)\sin(mx)\,dx = \pi\delta_{nm}$$

Homework

Same logic for $n \neq m$ with the sines also.

Orthogonal

$$\int_{-\pi}^{\pi} f(x)\,dx = \int_{-\pi}^{\pi} A_0\,dx + \sum_{m=1}^{\infty}\left[a_m \underbrace{\int_{-\pi}^{\pi}\cos(mx)\,dx}_{0} + b_m \underbrace{\int_{-\pi}^{\pi}\sin(mx)\,dx}_{0}\right]$$

same p trick $(p \neq 0)$

O3. Fourier Series

$$f(x) = A_0 + \sum_{m=1}^{\infty}\left[a_m \cos(mx) + b_m \sin(mx)\right]$$

$$A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

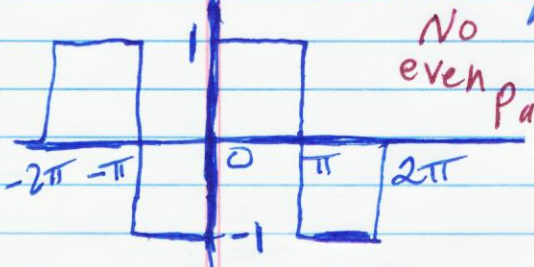
Use $f(x) = \dfrac{a_0}{2} + \displaystyle\sum_{m=1}^{\infty}\left[a_m \cos(mx) + b_n \sin(mx)\right]$

Due to πS_{nm}

So $a_0 = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\,dx \quad a_n = \dfrac{1}{\pi}\int_{-\pi}^{\pi}\cos(nx) f(x)\,dx \quad b_n = \dfrac{1}{\pi}\int_{-\pi}^{\pi}\sin(nx)\,dx$

all have the nice $\frac{1}{\pi}$

O4. The Square Wave



Sketch

These integrals will give zero.

No even part in series $\left\{ a_0 = 0 \quad a_n = 0 \right.$ odd function (see sketch)

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$$

Note:
$f(x) = 1$ 2nd half
$f(x) = -1$ 1st half

$$b_n = \frac{2}{\pi}\int_{0}^{\pi}\sin(nx)\,dx$$

$$b_n = \frac{2}{\pi}\left[-\frac{\cos(nx)}{n}\right]_0^{\pi} = -\frac{2}{\pi}\frac{1}{n}\left[\cos(n\pi) - \underbrace{\cos(0)}_{1}\right]$$

For $n = 1, 3, 5 \ldots \quad \cos(n\pi) = -1$

If n is even, you get 1 minus 1

$$\Rightarrow \frac{4}{\pi}\frac{1}{n} \quad \text{odd}$$

$$f(x) = \frac{4}{\pi}\left[\sin x + \frac{1}{3}\sin(3x) + \frac{1}{5}\sin(5x) + \ldots\right]$$

← Fourier 16 app →

M. J. Ruiz, "Free sixteen harmonic Fourier series web app with sound," Phys. Educ. 53, 025008 (March 2018).