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Orthogonality relation for covariant harmonic-oscillator wave functions

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Orthogonality relations for the Kim-Noz covariant harmonic-oscillator wave functions are discussed. It is shown that the wave functions belonging to different Lorentz frames satisfy an orthogonality relation. Furthermore, it is shown that for $n = m$ transitions there is a contraction factor of $(1-\alpha^2)^{(n+1)/2}$, where α is the velocity difference between the two Lorentz frames.

The covariant harmonic-oscillator wave functions recently proposed by Kim and Noz [1] can be applied to a wide range of hadronic processes. The harmonic-oscillator characteristics prominently show up in hadronic mass spectra. The Lorentz-contraction properties of the oscillator wave functions can be seen in the nucleon elastic form factors and other electromagnetic transition amplitudes. [2]

In their paper, Kim and Noz are primarily concerned with the probability interpretation of their covariant wave functions. Their oscillator wave functions satisfy all the requirements of nonrelativistic quantum mechanics and enable us to extend the probability concept to the relativistic region. Kim and Noz, however, did not explicitly calculate the overlap integral of harmonic-oscillator wave functions belonging to two different Lorentz frames. The purpose of this note is to perform this overlap integral.

Kim and Noz start with the following differential equation:

$$\frac{1}{2}\Big\{-\nabla^2+\frac{\partial^2}{\partial t^2}+\omega^2[\vec{x}^{\,2}-t^2]\Big\}\psi(x)=\lambda\psi(x). \qquad (1)$$

This harmonic-oscillator equation is separable in the $\vec{x}$ and t variables. Kim and Noz then observe that the above equation can also be written as

$$\frac{1}{2}\Big\{-\nabla_y^2+\frac{\partial^2}{\partial y_0^2}+\omega^2[\vec{y}^{\,2}-{y_0}^2]\Big\}\psi(y)=\lambda\psi(y), \qquad (2)$$

where the y variables are the Lorentz transforms of the x variables:

$$\begin{aligned} &y_1=x_1,\quad y_2=x_2,\\ &y_3=(1-\beta^2)^{-1/2}(x_3-\beta t),\\ &y_0=(1-\beta^2)^{-1/2}(t-\beta x_3). \end{aligned} \qquad (3)$$

Equation (2) is also separable in the y variables. The normalizable solutions in the y variables are the Kim-Noz wave functions. Their wave function has the form

$$\begin{aligned}\psi_\lambda(y)=&NH_{n_1}(y_1)H_{n_2}(y_2)H_{n_3}(y_3)\\ &\times exp[-\tfrac{1}{2}\omega(\vec{y}^{\,2}+y_0^2)],\end{aligned} \qquad (4)$$

where $\lambda=\omega(n_1+n_2+n_3+1)$, and N is the normalization constant. The elimination of timelike oscillations can be done covariantly. [1] Since the transverse oscillations do not undergo Lorentz transformations, we shall assume that $n_1=n_2=0$ in the following discussion.

The purpose of this note is to evaluate the following integral:

$$T_{nm}(\beta,\beta')=\int\psi_n(y)\psi_m(y')d^4x, \qquad (5)$$

where y' variables are the y variables of Eq. (3) with β'. We can evaluate the above integral using the generating function for Hermite polynomials. This generating function has the form

$$exp(-s^2+2sy_3)=\sum_{n=0}^{\infty}\frac{s^n}{n!}H_n(y_3). \qquad (6)$$

We can now use the integral

$$\begin{aligned}I(s,r)=\Big(\frac{\omega}{\pi}\Big)^2\int &d^4x\; exp(-s^2+2sy_3)exp(-r^2+2ry_3')\\ &\times exp[-\tfrac{1}{2}\omega(\vec{y}^{\,2}+{y_0}^2+\vec{y}'^{\,2}+{y_0'}^2)]\end{aligned} \qquad (7)$$

to evaluate the integral in Eq. (5). Both y and y' are functions of $\vec{x}$ and t. The transverse integrals can be performed trivially. For the t and z integrals we can use the variables ξ and η defined as

$$z=\frac{1}{\sqrt{2}}(\xi+\eta),\quad t=\frac{1}{\sqrt{2}}(\xi-\eta) \qquad (8)$$

to complete the square of the exponent of the Gaussian factor. We obtain

$$I(s,r)=(1-\alpha^2)^{1/2}exp[2rs(1-\alpha^2)^{1/2}], \qquad (9)$$

where $\alpha=(\beta-\beta')(1-\beta\beta')^{-1}$. Since the power-series expansion of Eq. (9) contains only equal powers of r and s, the integral of Eq. (5) vanishes unless $m=n$. When $m=n$, we find from the coefficient of $(rs)^n$ in the expansion of Eq. (9)

$$T_{nn}(\beta,\beta')=(1-\alpha^2)^{(n+1)/2}. \qquad (10)$$

The above results can be expressed in the following orthogonality relation [3]:

$$T_{nm}(\beta, \beta') = (1 - \alpha^2)^{(n+1)/2} \delta_{nm}. \tag{11}$$

The above result suggests that the covariant harmonic oscillators behave like nonrelativistic oscillators if they are in the same Lorentz frame. If two oscillators are in different frames, the orthogonality is preserved. The ground-state oscillator wave function, consisting of one half-wave, is contracted by $(1 - \alpha^2)^{1/2}$. The nth excited state, consisting of $(n + 1)$ half-waves, is contracted by $(1 - \alpha^2)^{(n+1)/2}$.

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[1] Y. S. Kim and M. E. Noz, *Phys. Rev.* D **8**, 3521 (1973).

[2] R. Lipes, *Phys. Rev.* D **5**, 2849 (1972). Lipes calculates his transition amplitudes in the Lorentz frame where the excited-state resonance is at rest, and his wave functions coincide with those of Kim and Noz in this particular frame. For this reason, the use of the Kim-Noz wave function will lead to Lipes's calculation.

[3] The first attempt to get the Lorentz contraction factor was made by Markov, who obtained a similar result for $n = 0$ and $\beta' = 0$. See M. Markov, *Nuovo Cimento Suppl.* **3**, 760 (1956).