Kepler's Third Law Without a Calculator

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At the University of North Carolina at Asheville (UNC-A) we have used Kepler's third law in a liberal-arts conceptual astronomy course to help students sharpen their quantitative skills without using a calculator. Doing quantitative physics without a calculator represents one of the many ways we can study the physical world. Furthermore, it is fun.

Clever reasoning to arrive at rough answers was popularized by the legendary Enrico Fermi.[1] The resulting quick estimates or back-of-the-envelope calculations serve as an excellent complement to the detailed analysis of data and theory. We are all familiar with how "too often the somewhat awkward numbers obscure the physics,"[2] and that at times we need to "shift student focus onto the physical processes and away from their calculators."[3]

Our students do get a chance in lab to work with Kepler's third law and precise data, using a spreadsheet. The standard linear graph is obtained when the cube of the semimajor axis is plotted against the square of the period.[4] Other authors have developed activities with Kepler's third law involving graphing calculators,[5] spreadsheets,[6] techniques with graph paper,[7] interactive tools,[8] and a new formulation of the law.[9] The art of estimation adds to the arsenal of these innovative approaches.

Kepler's Third Law and the Planets

Kepler's third law can be stated for circular orbits as

$$R \times R \times R = T \times T,$$

where R is the distance from the Sun in astronomical units (AU) and T is the time to complete one orbit in years. The Sun is taken to be infinitely massive compared to the mass of the orbiting body so that the center of mass is at the center of the Sun. For elliptical orbits, R is the average of the closest distance from the Sun (R_{min}, orbiting body at perihelion) and the farthest distance (R_{max}, orbiting body at aphelion). This description for R is more intuitive for the general student than "semimajor axis." In most of our examples we focus on finding T, given R.

Our UNC-A astronomy course has no prerequisites. If you can balance your checkbook, you know enough math to take our course. We have fun with the trivial case to break the ice in a large-class setting as we proceed into quantitative areas. For Earth, we have $R = 1$ AU. Therefore, $R \times R = 1$ and $R \times R \times R = 1$. We then ask students for the number that when multiplied by itself gives 1. They find the question humorous, becoming more at ease. The answer provides us with T since $T \times T = 1$ from Kepler's third law. Of course the result for the period is $T = 1$ year.

Our first "real" example is Jupiter. For Jupiter, $R = 5$ AU. Therefore, $R \times R = 25$ and $R \times R \times R = 125$. Which number when multiplied by itself gives 125? Since the value 125 doesn't appear as a perfect square in the standard multiplication table, we tell students it seems that we are at an impasse with no calculator handy. So we improvise, nudging the value to the nearest perfect square, which is 121. The square root is then $T = 11$ years. The more accurate value

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R = 5.2 AU gives 11.9 years with a calculator.

Pasachoff does a similar analysis[10] in reverse. He begins by stating that "astronomers are often content with approximate values that can be calculated in your head." He then provides a description of how to arrive at R from T by first pointing out that astronomical observations of Jupiter indicate that Jupiter takes T = 11.86 years to orbit the Sun. Rounding off, one gets $T \times T = 12 \times 12 = 144$. A trial-and-error method with cubes leads to R being a little over 5 AU as the best approximate answer.[10]

For Saturn, R is roughly 10 AU. Therefore, $R \times R \times R = 1000$, which among friends is 900. If a friend owes you \$1000, wouldn't you take \$900 as being close enough? Not all students agree to that, but with the approximation $T \times T = 900$, we arrive at T = 30 years. Or, using T = 33 years since $33 \times 33 = 1089$ (essentially our original 1000), we are still led to the rounded-off value of T = 30 years. Saturn's actual period is 29.5 years with R = 9.5 AU.

We conclude our class by asking students to close their eyes and participate in a blindfold mental calculation. Suppose an asteroid or spacecraft orbits the Sun at R = 4 AU. What is the period T? First, we ask what is $R \times R$? Students readily answer 16. Then we ask what is $R \times R \times R$, i.e., 16×4? Though perhaps not as fast, they still arrive at 64. Finally, we ask which number times itself gives 64. Students easily answer 8 and arrive at the period of 8 years.

We then commend our introductory students, many of whom fear math. We tell them they just applied Kepler's third law in their heads, a feat that requires a series of quantitative steps applying a mathematical formula. Praising them on their achievement builds their confidence in quantitative reasoning. It also helps to strengthen the rapport between the students and the appreciating instructor.

Though the following points are beyond the scope of a general-education class, the teacher might become more confident in understanding these subtleties. The average distance R is not only the average of the perihelion and aphelion distances, but also the path-average distance.[11] The path-average distance is not the same as the time-average distance, or for that matter, the average found by integrating with respect to the angle variable.[9,12]

Kepler's Third Law and Halley's Comet

The most famous of comets, Comet Halley,[13] has a period of 76 years. Slight variations in the period occur due to the locations of planets (such as Jupiter and Saturn) as Halley makes its eccentric orbit around the Sun. From Kepler's third law we can estimate R. We have $T \times T = 76 \times 76 \approx 75 \times 75 = 25 \times 3 \times 25 \times 3 = 25 \times 25 \times 9 \approx 27 \times 27 \times 8$, which gives a cube root $R = 3 \times 3 \times 2 = 18$ AU. The subtle last steps, i.e., nudging the factors of 25 to 27 and replacing 9 with 8, are necessary so we can take a cube root in our heads. The actual value for Halley's comet is 17.9 AU.

We can also determine beyond which planet Halley reaches at aphelion. But we have to remember that Halley's comet is very elliptical. It is unlike a planetary example where the planet's minimum and maximum distances from the Sun are nearly the same. Here we need to emphasize that the R in Kepler's third law is $R = (R_{min} + R_{max})/2$. Therefore, $R_{max} = 2R - R_{min}$. Taking Halley at perihelion to be approximately 1 AU from the Sun gives the largest distance to be R_{max} = 2 × 18 AU – 1 AU = 36 AU – 1 AU = 35 AU. This places the aphelion beyond Neptune's orbit. The actual minimum and maximum distances for Halley's comet are 0.6 AU and 35.3 AU, respectively.

Space Travel and Least-Energy Orbit

Imagine a group of scientists and engineers at a meeting in the 1960s discussing space flights to planets. The director of the meeting proposes a trip to Jupiter and asks for an estimate of travel time. There is silence until one person suggests a small group should go investigate it. Meanwhile, another turns over an envelope to show a diagram with an estimate of about two years. This contrived scenario illustrates the power of a very clever application of Kepler's third law, described below.

Unlike science fiction stories, we cannot easily direct a spacecraft over large distances by firing super rockets. A simple trajectory to Jupiter uses the Earth's motion to help get the spacecraft traveling in an elliptical orbit around the Sun with the beginning of the voyage at the perihelion (R = 1 AU) and the destination Jupiter at the aphelion (R = 5 AU). The launch has to be timed so Jupiter will be in the right place at the proper time. The path we have described is also called a least-energy orbit or Hohmann transfer orbit.[14]

The period of the elliptical orbit is found using Kepler's third law with the average distance: $R = (1 + 5)/ 2 = 3$ AU. Then, $R \times R \times R = 27 \approx 25$, which gives $T = 5$ years. Since the spacecraft makes one-half of the complete orbit, the time is $T = 2.5$ years. Using $R = 5.2$ gives the more precise $T = 5.5$ years and a travel time of 2.8 years.[15] As an example, Voyagers I and II left Earth in 1977 and arrived in 1979.

However, the Voyager trajectories, as with the earlier Pioneer 10 (launched 1972) and Pioneer 11 (1973), exceeded least-energy orbital specifications and the travel times were less than two years. On the other hand, Galileo, launched in 1989, first made three orbits around the Sun, flying by Venus once and the Earth twice to gain speed. When a spacecraft swings by a planet to get a speed boost, the effect is called a gravity assist. After the Venus-Earth-Earth Gravity Assist (VEEGA), the Galileo spacecraft began essentially a Hohmann transfer orbit from Earth (December 1992) to Jupiter (December 1995), taking slightly more than the calculated 2.8 years for an Earth-Jupiter Hohmann transfer. A nice additional problem is to analyze the travel time for the Cassini-Huygens spacecraft to Saturn (arrival date: July 2004). An analysis is posted online.[16]

Here are two challenging Fermi questions. The first is to estimate the distance above the Earth for a geosynchronous satellite. The geosynchronous problem is often found in an introductory physics course with mathematics.[17] The second challenging Fermi question is to estimate the period of a low-altitude satellite.[18] Solutions are posted online.[19]

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11. A way to visualize this is to draw the usual ellipse with a string looped around the foci with a taut pull. By definition of the ellipse, the average of the lengths of the two string sections from each focus to a point on the ellipse is equal to the semimajor axis. Consider these as "conjugate" lengths. Then, each point on the planet's orbit can be paired up with a "conjugate" point elsewhere on the orbit. The average of these two conjugate distances from the Sun (left focus) is always equal to the length of the semimajor axis.
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16. See EPAPS Document E-PHTEAH-42-010409. This document may be retrieved via the EPAPS homepage (http://www.aip.org/pubservs/epaps.html) or from ftp.aip.org in the directory/epaps/ in the phys_teach folder. See the EPAPS homepage for more information.
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