Boomwhackers and End-Pipe Corrections

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End-pipe corrections seldom come to mind as a suitable topic for an introductory physics lab. Yet, the end-pipe correction formula can be verified in an engaging and inexpensive lab that requires only two supplies: plastic-tube toys called boomwhackers[1] and a meterstick. This article describes a lab activity in which students model data from plastic tubes to arrive at the end-correction formula for an open pipe. Students also learn the basic mathematics behind the musical scale, and come to appreciate the importance of end-pipe physics in the engineering design of toy musical tubes.

Goals

Authors have addressed measuring frequencies of corrugated tubes[2] and PVC pipes.[3-6] Some of these authors[2,5,6] incorporate end-pipe corrections in their discussions. The majority of prior literature has focused on corrugated tubes or plumbing pipes. In contrast, this paper focuses on the engineering design of a set of eight commercially available tubes, the colorful and fun boomwhackers, which produce musical tones. The set of eight without the holder, case, and end cap is available for as low as $19.95 as of the writing of this paper. See Fig. 1 for a photo of a student measuring the length of a boomwhacker.

Fig. 1. Nicole Munger measuring the length of a boomwhacker.

The goals for student learning in our boomwhacker lab are:

1) to make accurate length measurements to the nearest millimeter to calculate the end-pipe correction,
2) to learn about the equal-tempered and just diatonic musical scales, and
3) to appreciate the importance of a rather advanced, yet simple formula in the engineering design of a commercial product.

Background theory

The resonance formula with end-pipe corrections for an open pipe employs an effective pipe length $L' = L + X$, where L is the actual length of the pipe and X is the end-pipe correction. The end correction is due to the fact that the pressure nodes are outside the pipe, extending the pipe length by approximately $0.6r$ for each open end, where r is the inner radius of the pipe.[7] Many authors[2,5,6,8] report the value $0.61r$ as given by the model of Levine and Schwinger.[9] The end-pipe correction is valid for wavelengths much greater than the circumference of the pipe.[10] By the way, the second coauthor of Ref. 9 is Julian Schwinger of quantum electrodynamics (QED) fame.[11]

In our lab we model the end-pipe correction as er for each open end and determine the coefficient e from our data. Therefore, the effective length L' of each pipe is found by adding an extension $X = 2er$ (two open ends) to the actual physical length L:

$$L' = L + X = L + 2er. \tag{1}$$

As part of the lab activity, the students need to understand equal temperament. See Fig. 2 for the simple one-octave keyboard consisting of the eight notes of the major scale (the white keys) and the five black keys (shaded).

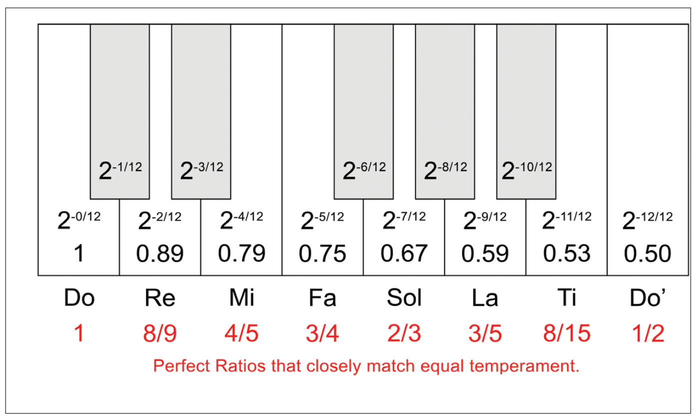

Fig. 2. Relative lengths of pipes to produce the major scale (white keys): Equal-tempered values (with powers of two) and closest whole-number ratios.

The ratio for the pipe lengths in going from Do to Do', the octave, is set to be 1/2. Stepping from the first note Do to Do' requires 12 steps as you hop from each note to its adjacent right neighbor. Thus, for each of the 12 steps we use the 12th root of 1/2. For frequency, we take the inverse, i.e., the 12th root of 2 since frequency is inversely proportional to wavelength.

As part of the preparatory background before doing the lab measurements, I pick up boomwhackers and start smacking them against my head. This display enhances student interest and the students quickly see that the shorter tubes produce higher pitches. By eyeballing the length of a boomwhacker alongside the longest boomwhacker (Do), the students can even obtain an understanding of the musical scale defined by simple ratios. You can estimate cases where the shorter pipes are roughly 1/2, 2/3, 3/4, and 4/5 the size of the Do-boomwhacker. The corresponding notes of the shorter pipes are the higher Do (an octave higher, which we call Do'), then Sol, Fa, and Mi, respectively.

The nearest whole-number ratio for each pipe length relative to the reference Do is given in red below each note name in Fig. 2. You can arrive at the whole-number ratios by matching their decimal equivalents with the equal-tempered values, e.g., 4/5 is close to the equal-tempered 0.79. The ratios shown here are those for the just scale or just intonation, a modification of the Pythagorean scale.[3] Whole-number ratios for the black keys are discussed by LoPresto.[12]

The procedure and experimental results

You can pass around tubes so that students can take turns making measurements. This approach stretches the use of one set of boomwhackers for many students. Students can work either alone or in pairs. These options make the boomwhacker lab extremely inexpensive and very flexible.

For the laboratory activity students measure the lengths of the open boomwhacker pipes. For an open pipe, neglecting end effects, the fundamental resonance occurs for wavelength

$$\lambda = 2L . \qquad (2)$$

We first want to ignore end effects and show that the ratio

$$R = \frac{L}{L_{\mathrm{Do}}} \qquad (3)$$

does not fare well when compared to the equal-tempered values R_{et}. Three local high school teachers,[13] three university faculty,[14] and one physics major in teacher education[15] took the data that is averaged in Table I. The average results are close to individual measurements since it is easy to measure to the nearest millimeter (three significant figures) and we are reporting ratio results to two significant figures. Note that as the pipes get shorter, deviations from R_{et} increase since the end corrections become more important. This motivates us to find a better model for the data.

We model the end corrections using effective pipe lengths given by Eq. (1):

$$\mathrm{R}_{\mathrm{et}} = \frac{L'}{L_{\mathrm{Do}}'} = \frac{L+X}{L_{\mathrm{Do}}+X}, \qquad (4)$$

where $X = 2er$ since we have an end correction er for each open end. We solve for L as we intend to plot L as a function of R_{et} and then use a least-squares fit to determine X and L_{Do}. We find

$$L = (L_{\mathrm{Do}} + X)R_{\mathrm{et}} - X. \qquad (5)$$

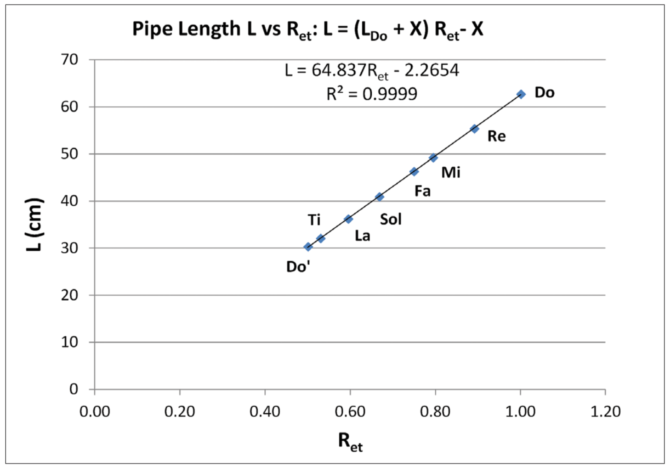

Fig. 3. Least-squares fit for plot of L vs R_{et} incorporating end corrections.

Table I. Pipe length (L), $R = L/L_{\mathrm{Do}}$, and the ratio R_{et} calculated from equal temperament, neglecting end corrections. Discrepancies increase as end corrections become more important for the shorter pipes.

Note	L (cm)	R	R_{et}
Do	62.7	1.00	1.00
Re	55.4	0.88	0.89
Mi	49.2	0.78	0.79
Fa	46.3	0.74	0.75
Sol	40.9	0.65	0.67
La	36.2	0.58	0.59
Ti	32.1	0.51	0.53
Do'	30.3	0.48	0.50

A least-squares fit is plotted in Fig. 3 using Excel. The best linear fit is

$$L = (64.837)R_{\mathrm{et}} - 2.2654 \qquad (6)$$

with a correlation of $R^2 = 0.9999$, indicating excellent agreement with our linear modeling equation. The end-pipe coefficient e is found from $X = 2.2654 = 2er$. Using the measured value $r = 2$ cm for the inner radius, $e = 0.57 = 0.6$. Our result is consistent with the value $0.6r$ given in the literature for the end-correction factor.

For the longest pipe, we have $R_{\mathrm{et}} = 1$ and our best fit gives $L_{\mathrm{Do}} = 64.837 - 2.2654 = 62.6$ cm. This is in agreement with our measured value of 62.7 cm since our measurements have an experimental uncertainty of about 0.1 cm. By the way, if one uses the effective lengths $L' = L + 2(0.6)r$, then the experimental ratios similar to those found in Table I match the equal-temperament ratios R_{et}.

Conclusion

In summary, this lab activity shows that engineers need to consider end corrections in their design of boomwhacker tubes. When end corrections are included, one can determine the proper lengths to manufacture the toy tubes so that they produce the correct pitches for the major scale. This is a lab with easy measurements made to the accuracy of 1 mm for tubes that vary in lengths from about 30 cm to 60 cm.

Rather than take the simple approach of checking the end-pipe correction formula where the student is given the $0.6r$ term, we explored a model with end-pipe correction er for each open end. We then verified our model by analyzing our data with a spreadsheet. In this way, we not only validated the model, we also found the coefficient e to be 0.6.

One can also extend the lab activity by measuring frequencies using techniques already discussed in the literature.[2,4-7] Furthermore, one can purchase end caps[16] called "octavators" to demonstrate that the pitches drop an octave when an open pipe is closed on one end. Though a student can cover one end of a boomwhacker with the palm of one's hand while a lab partner taps the tube, it is nice to have the end caps, especially since students find the name "octavator" captivating. The name also helps one remember the physics!

Acknowledgments

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References

1. The Diatonic Set of 8 Boomwhackers by Whacky Music is available from vendors that you can easily find at www.amazon.com. Be sure to purchase the eight tubes for the major scale. Some pipe sets come with only five tubes. More expensive sets include a bag, stand, and a single end cap.
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4. Bob Brazzle, "Inexpensive instruments for a sound unit," *Phys. Teach*. **49**, 228–230 (April 2011).
5. Michael E. Bacon, "Speed of sound versus temperature using PVC pipes open at both ends," *Phys. Teach.* **50**, 351–353 (Sept. 2012).
6. M. E. Bacon and Steven Torok, "An experimental investigation of the end effects for Blue Man Group® pipes," *Phys. Teach*. **49**, 152–154 (March 2011).
7. Daniel R. Raichel, *The Science and Applications of Acoustics,* 2nd ed. (Springer, New York, 2006), p. 135.
8. N. H. Fletcher and T. D. Rossing, *The Physics of Musical Instruments,* 2nd ed. (Springer, New York, 1999). Thomas D. Rossing, F. Richard Moore, and Paul A. Wheeler, *The Science of Sound,* 3rd ed. (Addison-Wesley, San Francisco, 2002), pp. 65-66.
9. Harold Levine and Julian Schwinger, "On the radiation of sound from an unflanged circular pipe," *Phys Rev.* **73** (4), 383–406 (1948).
10. Levine and Schwinger give $0.6133r$ as the correction for each open end when $kr \equiv \frac{2\pi r}{\lambda} << 1$. From their nice plot of end correction versus kr, one finds that the end correction is a slowly decreasing function over the range of pipe lengths of interest.
11. Julian Schwinger shared the Nobel Prize in 1965 with Richard Feynman and Sin-Itiro Tomonaga for their independent work on quantum electrodynamics (QED).
12. Michael C. LoPresto, "Fourier analysis of musical intervals," *Phys. Teach*. **46**, 486–489 (Jan. 1994). See Table I for a complete set of whole-number ratios for the 12-tone scale (which includes the five black keys).
13. Mike Bowman from North Buncombe High School, Wayne Hamlin from T. C. Roberson High School, and Biff Spisak from A. C. Reynolds High School.
14. Judy Beck, James Perkins, Department of Physics, UNC Asheville, and the author.
15. Nicole Munger, a graduate with an associate's degree from Asheville Buncombe Technical Community College and a physics major at UNC Asheville, working in the comprehensive science teaching licensure program.
16. End caps, called "octavators," are nice to have and can be purchased separately. Then you don't have to hold your hand over one end to form a closed pipe. Also, the "octavator" sounds cool!

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