Lissajous Figures, Musical Intervals, and the Just Diatonic Scale

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The frequency ratios for the just diatonic scale are obtained by identifying musical intervals corresponding to Lissajous figures. The demonstration integrates the engineering physics of Lissajous patterns with the recognition of musical intervals through simple ear training. A free HTML5 app has been developed for this class activity and the program can be run online or downloaded to the desktop. A video[1] is provided illustrating the use of the software.

Background

Discussion of musical scales can be found in an early issue (1934) of *The American Physics Teacher*[2] and a recent issue (2015) of its successor journal, the *American Journal of Physics*.[3] Both references include discussion of the equal-tempered tuning we use today as well as tunings based on the perfect integer ratios described in this paper. These tunings are listed in Table I, where the first note in the major scale has been set to 240 Hz. For years I have used 240 Hz for the first degree of the major scale since all the frequencies with integer ratios work out easily as whole-number frequencies without a calculator. Reference 2 lists the relative frequencies as whole numbers where 24 is assigned to Do (the note C), a procedure that was made by the legendary Helmholtz as far back as 1895 in his masterpiece *On the Sensations of Tone as a Physiological Basis for the Theory of Music*.[4]

Table I. The just major scale with Ratios and the equal-tempered scale. The movable Do convention (widespread in the United States and United Kingdom) is used, where Do can be assigned any frequency.

Note	1	2	3	4	5	6	7	8
Name	Do	Re	Mi	Fa	Sol	La	Ti	Do'
Frequency Ratio	1:1	9:8	5:4	4:3	3:2	5:3	15:8	2:1
Just Intonation *f* (Hz)	240	270	300	320	360	400	450	480
Equal Tempered *f* (Hz)	240.0	269.4	302.4	320.4	359.6	403.6	453.1	480.0

Be aware that there are two common conventions for teaching voice: the fixed-Do solfège system and the movable-Do solfège.[5] Movable-Do allows Do to be assigned any pitch (frequency). In Table I, Do has been assigned 240 Hz. On the other hand, fixed-Do, which is used in many parts of the world, always assigns Do to middle C. Middle C has a frequency of 261.6 Hz in equal-tempered tuning where middle A is set to 440 Hz. For those that prefer the fixed-Do convention, focus on the degrees of the scale 1, 2, 3, etc. rather than the names Do, Re, Mi, etc.

The major scale tuned to whole-number ratios relative to the first note in the scale is called the just diatonic scale or major scale with just intonation. The scale presented here is the most well-known just scale and is also known as Ptolemy's intense diatonic scale,[3] a perfect Ptolemaic Sequence,[6] or the syntonic diatonic.[7] "Ptolemy's intense diatonic is the basis for Western European just intonation."[8] However, Ptolemy's just scale had to be later rediscovered in the early Renaissance and championed by medieval theorists such as Folgiano (1529) and Zarlino (1558).[7]

I recommend against showing Table I to your students first as we will discover the main results (the row labeled "just intonation") of the table through the activity described in this paper. I come back later in the semester for the equal-tempered part of Table I. The aim of this paper is to arrive at the closest small whole-number frequency ratios for notes in the major scale from an experiment that combines physics and music techniques. Each frequency ratio to be measured is between a note in the scale and the reference Do. Such a pair of notes spans a musical interval. No prior knowledge of intervals is required. Each interval will be defined in this paper and the experiment will lead to the ratios of integers. The most pleasing intervals (consonance) are those with ratios of small integers, while the most dissonant (dissonance) are those with ratios of larger integers. LoPresto has a nice paper in this journal treating consonance and dissonance at some length.[9] The search for pleasant intervals historically led to the development of the just diatonic scale and its variations.

Equal temperament

This paper focuses on the just diatonic scale, which I introduce early in my physics of sound and music course. Though equal temperament can be covered later, the instructor should point out that the tuning system used today does not consist of exact integer ratios except for the octave. Equal temperament evolved since the integer ratios of just scales are not preserved if one starts playing the scale at a different starting point, a process called transposition in music.

Equal temperament deals with the full 12-tone scale. The major scale in the key of C consists of seven distinct white keys plus the white key Do' that completes the scale and begins the scale an octave higher. When the five black keys are added to the seven distinct pitches, we have the 12-tone or chromatic scale. There are 13 notes when Do' is included, with 12 semitone steps from Do to Do'. Equal temperament requires that the last note Do' in Table I be twice the frequency of the first note Do. This system will give the best compromise for playing scales in different keys (different starting points) with results fairly close to the just diatonic scale, as shown in Table I.

For an analogy with investing money, consider 13 annual dates (e.g., 2000, 2001, … 2012) that span a time period of 12 years. What multiplying factor applied annually is needed for money to double in 12 years? The answer is the 12th root of 2, i.e., $(2)^{(1/2)} = 1.05946...$, which corresponds to an interest rate of about 5.9% applied once every year. Pick any arbitrary frequency and start multiplying by the 12th root of 2 and the equal-tempered frequencies emerge.

Reference 2 has an excellent table giving all 13 notes in one octave of the chromatic scale with their just ratios and comparisons to equal-tempered decimal equivalents. The musical intervals that are introduced later in this paper are also included in Refs. 3 and 9. In an earlier paper of mine, I provide[10] a figure of a keyboard where each key is labeled with the appropriate power of the 12th root of 2 to describe relative wavelengths, the inverse of the frequencies via $v = \lambda f$.

The plan for deriving integer frequency ratios for the major scale

In physics courses the usual practice is to list the frequency ratios of the just scale from reference sources as in Table I above. Instead, we will arrive at these ratios through a demonstration where both physics and music play an essential role. The basic technique is to first find a Lissajous pattern and then identify the musical interval formed by the two corresponding tones with a simple ear-training musical technique.

First, students are introduced to Lissajous figures. Second, students are given a short ear-training session, and third, the experiment begins using Lissajous figures and music. In an introductory physics course for non-science majors, all three topics can be presented in a 75-minute class period. The beauty of the demonstration is that the physics needs the music and the music needs the physics to arrive at the results.

Lissajous figures

I introduce Lissajous figures as a game where one moves on a board following instructions. The instructions are given by two graphs: the first being how to move horizontally (east-west) as a function of time and the other being how to move vertically (north-south) during the same time frames. I use triangle waves for the analysis so that the students can precisely plot the Lissajous figures and easily sketch them on graph paper due to the straight lines that result.

I first do two cases where the vertical frequency is the same as the horizontal frequency with phases chosen so that the Lissajous figures are a slanted line and diamond, respectively. You can find these cases described in the accompanying video.[1] For sake of brevity I proceed here to the third case I show in class. Figure 1 illustrates the set of instructions for which the vertical motion has twice the frequency of the horizontal motion and both waves start off from zero, first forming crests.

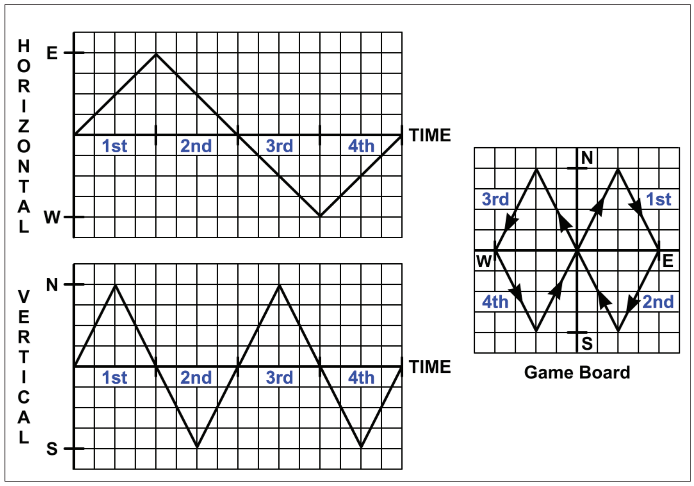

Fig. 1. Lissajous pattern for a vertical frequency that is twice the horizontal frequency where both the horizontal and vertical waves start from zero, forming crests.

The game board is to the right of the two instruction graphs. I first ask the students where we start on the game board. At time $t = 0$, the horizontal position is zero, i.e., neither east nor west. At the same initial time $t = 0$, the vertical position is also zero, i.e., neither north nor south. So we start at the center of the game board. The time frame for the game is broken up into four quarters like football or basketball. Each quarter of the game consists of four time blocks.

During the first quarter of the game, we must move east four blocks, while at the same time going north four blocks and then south four blocks. This path on the game board is labeled 1st. For the second quarter of the game, we must move back west to the center of the board while going south four blocks and coming back north by four blocks. See the label 2nd on the game board in Fig. 1. During the third quarter, we move west by four blocks while going north four blocks and then south four blocks.

By this time I ask students to switch from their analytical left brain to their artistic right brain and tell me what the fourth-quarter path should be to complete the artistic pattern. Their holistic thinking leads to completing the symmetry with the down-up path labeled in Fig. 1 as 4th. This answer is verified by referring to the instruction graphs.

The experimental importance of Lissajous figures lies in the ability to determine the frequency ratio of the vertical frequency to the horizontal frequency, $\frac{f_V}{f_H}$.

This technique will prove valuable in determining frequency ratios in our upcoming demonstration. I ask the students how many times does the pattern on the game board hit the ceiling. The answer is 2. How many times does the pattern touch the right wall? The answer is 1. Therefore, $\frac{f_V}{f_H} = 2$,

which leads to $f_V = 2f_H$, which we already know from inspecting the horizontal and vertical rules in Fig. 1. Later, the vertical frequency will be the unknown. The next part of the demonstration brings us to a music lesson.

Table II. The eight basic musical intervals and key reference songs.

Musical Interval	Interval Name	Key Reference Song
Do to Do	Unison	"America" or "God Save the Queen"
Do to Re	Second	"Do-Re-Mi"
Do to Mi	Third	"The Marines' Hymn"
Do to Fa	Fourth	"Here Comes the Bride"
Do to Sol	Fifth	"Twinkle, Twinkle, Little Star"
Do to La	Sixth	"My Bonnie"
Do to Ti	Seventh	"Superman Theme"
Do to Do'	Octave	"Somewhere Over the Rainbow"

Music ear training

The music ear-training lesson is an equally important ingredient for our goal. The ultimate aim of our experiment is to find beautiful Lissajous patterns, play the corresponding audio frequencies in each case, and identify the musical interval. How does one identify musical intervals? The method is a standard technique used by musicians that do not have perfect pitch, which means the vast majority of musicians.

When I was a graduate student in physics I was taking music courses with music majors at the University of Maryland. One day I expressed amazement at how musicians could identify intervals. A pianist-singer classmate, Sally Harmon,[11] taught me the trick. We prepare a reference list of songs that begin with each interval or where the interval is used prominently in the song. Then, when a teacher in an ear-training class plays two consecutive notes, the interval can be recognized from the relevant song that is associated with that interval. A set of songs for each of the eight intervals appears in Table II. The seventh is rare but a well-known example is the dramatic use of this interval in the theme for the movie "Superman" (1978) by John Williams, though it does not occur at the very beginning. Students readily recognize the seventh when I point it out to them[1] as they listen to the famous Superman theme.

See the accompanying video[1] for a demonstration of the ear-training exercise. After explaining this trick, students do quite well when I test them in class using a keyboard. I encourage them at home to have a family member punch out a C (white key Do) on a piano and another white key in the scale so they can impress everyone with their aural skills. But never state the song since that is our secret; always give the interval instead. They will astound piano students since piano teachers typically use up all teaching time with reading notes and there is no lesson time for aural skills. The third part of the demonstration is the actual experiment to measure the perfect frequency ratios for the just major scale, which comes next.

The experiment

In the old days I had to haul to class two oscillators, an oscilloscope, two amplifiers, two speakers, and a portable keyboard to do the experiment. Today I use a self-contained app[12] that I developed which can do it all. You may download the single HTML5 file, which includes all the source code. The single file contains the basic HTML, cascading style sheet (CSS) code, and the JavaScript for controlling the HTML5 audio API (Application Program Interface). A screenshot of the software app is shown in Fig. 2.

The app illustrated in Fig. 2 incorporates the equivalent of three oscilloscopes, two audio oscillators, and a major-scale keyboard, playing through the computer sound system. A student is asked to come to the front of the class and use the vertical slider (lower left oscilloscope display) to find a beautiful Lissajous figure. The horizontal oscillator is kept as a control variable fixed at 240 Hz. Using 240 Hz for "Do" allows the eight pitches of the just scale to come out as whole numbers. All the math can be accomplished without a calculator. I typically keep the sound muted at this stage so that the students can focus on the visual display in search for an esthetic pattern.

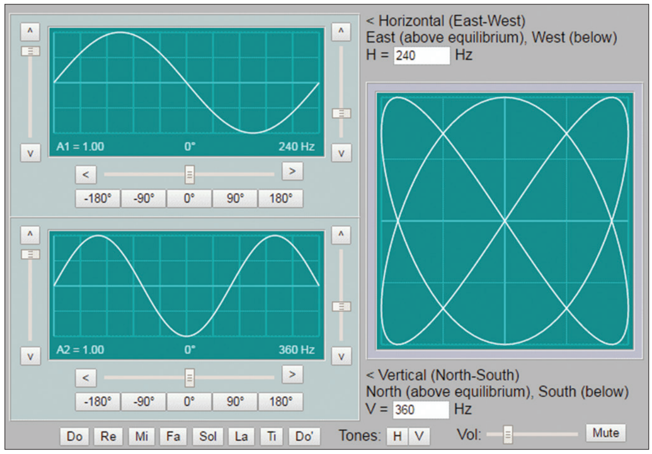

Fig. 2. Screenshot of software app to measure the frequency ratios of the just diatonic scale.

To direct the students somewhat, when the program first runs, the user is presented with a rotating elliptical shape, which suggests that the unison is near. Nudging the 241-Hz vertical frequency one hertz to equal the horizontal value of 240 Hz leads to a diagonal line, or a circle if the vertical wave is shifted by 90°. Always keep the horizontal wave fixed. Then I ask a student to find the next simplest case. This suggestion coaxes them to find the 2:1 Lissajous figure, which corresponds to the octave. In this way we find the frequency boundaries of the major scale: from 240 Hz to 480 Hz. To find the other notes in between, students restrict their additional searches to vertical frequencies between 240 Hz and 480 Hz. The experiment then goes much faster. I do not have the students do the Re or the Ti cases as these Lissajous figures have many lobes and we usually run out of time in class. But these lobes can be meticulously counted if ambitious students are interested and time permits.

Once a nice pattern is obtained such as the one shown in Fig. 2, students are asked how many times the pattern touches the ceiling. Students are reminded that the app uses sine waves instead of triangle waves so that the patterns are curved. But the same principle of touching the ceiling and right wall is applicable. Students readily determine that the frequency ratio in Fig. 2 is $f_H : f_V = 3:2$.

The next step is to turn on the audio. The buttons for the tones are labeled H and V. The H-button plays the horizontal tone and the V-button plays the vertical. These two tones toggle on and off as you press their associated buttons so you can hear the two tones simultaneously. By pressing H and then V, everyone hears the two tones in sequence. The interval is then determined from the reference songs listed in Table I. The answer is the fifth. I cannot sing, but I sing the tones anyway and my "Twinkle, Twinkle, Little Star" is good enough so that students recognize that the interval is indeed a fifth. A student in class most likely can be found who can do much better. If students incorrectly identify the interval as the fourth, try humming the audio tones heard in an attempt to hum "Here Comes the Bride." It will not work because the tones form a fifth rather than a fourth. Finally, one can always check with the tone keypad. The keypad notes also toggle on and off; however, the keypad is designed so you can only play one reference note at a time.

Though assigning 3:2 to the fifth is the most important conclusion, I calculate the frequency in hertz for our "Do" of 240 Hz. The frequency for the fifth is then

$$f_H = \frac{3}{2} f_V = \frac{3}{2} \cdot 240 \text{ Hz} = 360 \text{ Hz}.$$

This frequency can also be checked with the app, which lists the frequencies for each oscillator. In physics, there is always more than one way to accomplish anything. Our experimental results agree with the values shown in Table I, introduced at the beginning of this paper.

Conclusion

We have experimentally measured the frequency ratios of the major scale based on perfect ratios. The experiment is made possible by combining the engineering physics of Lissajous figures with recognition of the corresponding musical intervals from ear-training techniques encountered in music courses. Starting with Do = 240 Hz allows for simple mathematical calculations for all the frequencies that can be worked out easily on the board in class without a calculator. The major scale we use today is the equal-tempered scale, which allows for optimum consonances in different keys.[2,3, 6, 9]

During this demonstration I emphasize to my students that initially we search for beautiful patterns. The search is an esthetic one. We are confident that the search for beauty will lead to practical applications in science and music. Inspired by a comment that general relativist Larry Smarr[13] once made about Einstein's gravitational field equations, I tell my students that "in its best form, science and art are indistinguishable."

References

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10. M. J. Ruiz, "Boomwhackers and end-pipe corrections," *Phys. Teach.* **52**, 73–75 (Feb. 2014). See Fig. 2.
11. Sally Harmon, http://www.sallyharmon.com/, accessed February 10, 2017.
12. M. J. Ruiz, The Just Diatonic Scale App (2017), http://mjtruiz.com/TPT/diatonic/lissajous.html .
13. The original Larry Smarr quote is, "I think Einstein's Theory of Relativity is one of the most beautiful creations of human kind. It is both scientific and esthetic at the same time. It's one of the real moments in which beauty, in this case the mathematical beauty of equations, led to the science being discovered, and in its best form, science and art are indistinguishable." See http://archive.ncsa.illinois.edu/Cyberia/NumRel/Smarr_2.html (1995), accessed May 11, 2017.

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