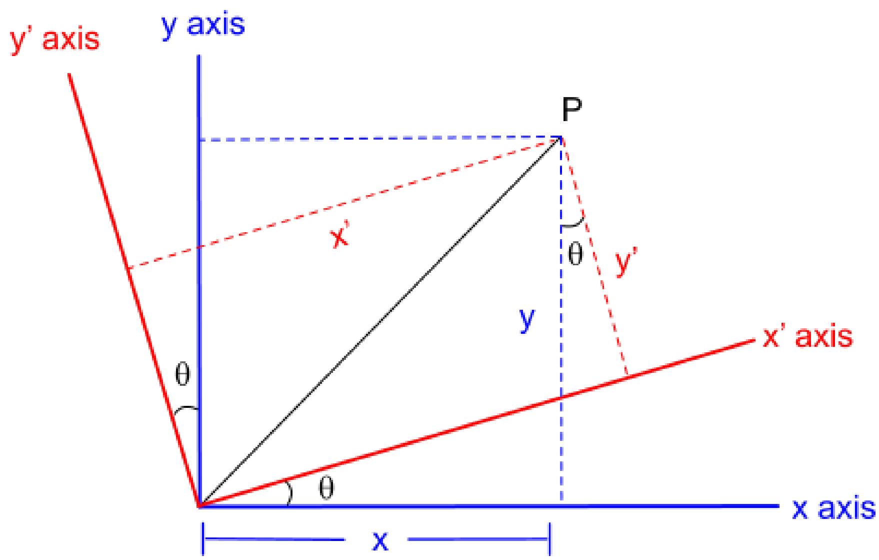
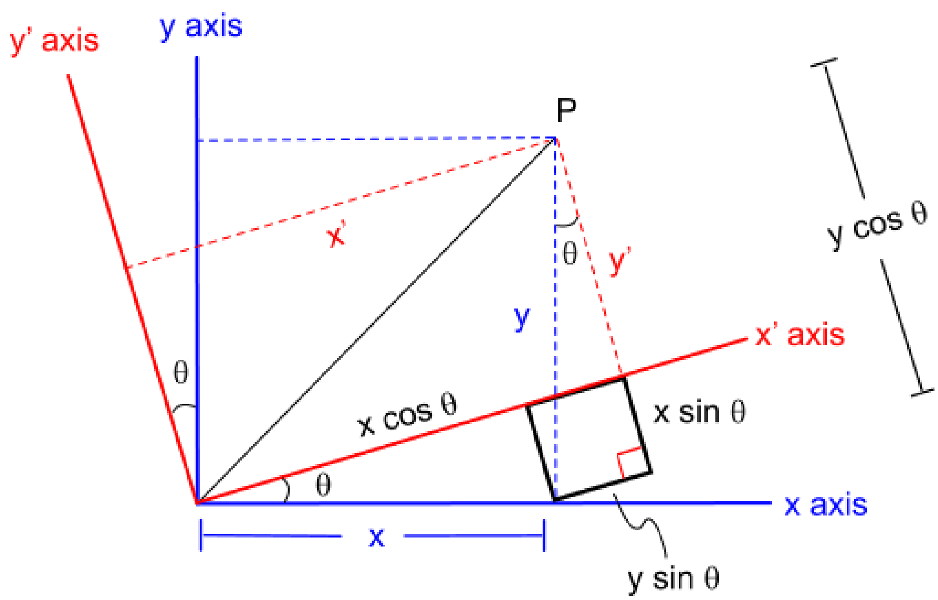
Electromagnetic Theory
Prof. Ruiz, UNC Asheville, doctorphys on YouTube
Chapter B Notes. Special Relativity

B1. The Rotation Matrix

There are two pairs of axes below. The prime axes are rotated with respect to the original x-y system.



The secret in relating (x', y') to (x, y) is to construct the cute rectangle you see in the figure.



$x' = x\cos\theta + y\sin\theta$ and $y' = y\cos\theta - x\sin\theta$.

$$x' = x\cos\theta + y\sin\theta$$
$$y' = -x\sin\theta + y\cos\theta$$

We can define the following matrix

$$R(\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}.$$

In matrix notation we can write

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = R(\theta)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}.$$

B2. Trig Identities

Now the fun. We will proceed to derive trig formulas in one step. Remember those days, perhaps in high school, when you first encountered complicated trig identities involving the sines and cosines of sums and differences of angles. Here you can derive these quickly. The combined rotation

$$R(\alpha+\beta) = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}\begin{bmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{bmatrix}$$

has to be equal to $R(\alpha+\beta) = \begin{bmatrix} \cos(\alpha+\beta) & \sin(\alpha+\beta) \\ -\sin(\alpha+\beta) & \cos(\alpha+\beta) \end{bmatrix}.$

Multiply the matrices and your a_{11} matrix element is your cosine identity for $\cos(\alpha+\beta)$. The element a_{12} takes care of $\sin(\alpha+\beta)$. The results are:

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$\sin(\alpha+\beta) = \cos\alpha\sin\beta + \sin\alpha\cos\beta.$$

Replace β with $-\beta$ and you arrive at the formulas involving the differences. Remember that the cosine is an even function, i.e., $\cos(-\beta) = \cos\beta$, and the sine is an odd function such that: $\sin(-\beta) = -\sin\beta$.

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$
$$\sin(\alpha-\beta) = -\cos\alpha\sin\beta + \sin\alpha\cos\beta.$$

This is an example of the "magic" of the rotation matrix. You might say these are Feynmanesque derivations. We get the result in a couple of lines, while the high school proof goes on and on with intricate diagrams and multiple algebraic steps that can take over a page.

In fact, we are not even afraid of the triple-angle sum, $\cos(\alpha+\beta+\gamma)$. Just multiply another matrix and pick off the appropriate part. This is another characteristic of Feynman - using theoretical techniques to do even more general and more difficult proofs with relative ease.

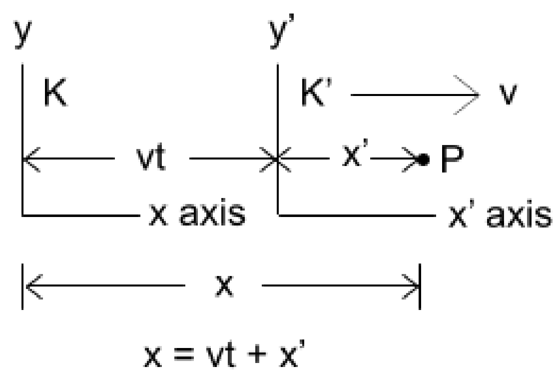
PB1 (Practice Problem). Use the above formulas to derive the result for $\tan(\alpha-\beta)$. Then replace β with $-\beta$ to arrive at the identity for $\tan(\alpha+\beta)$. Arrange your results to look like the standard forms:

$$\tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$$

and

$$\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}.$$

B3. Galilean Transformation



We consider inertial frames in this chapter. An inertial frame is one either at rest or moving in a straight line with constant speed. At the right are two such frames. The K frame can be considered at rest and the K' frame moving at speed v along the x-axis relative to the K frame.

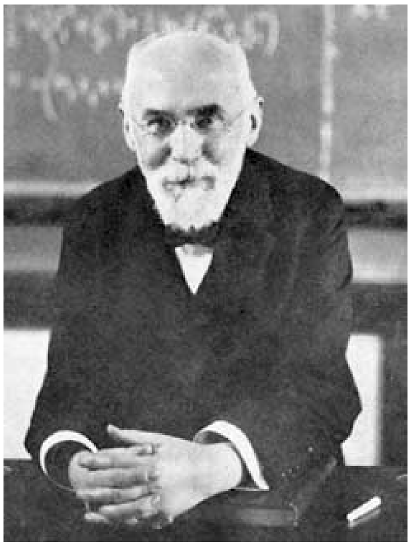
We can introduce the time coordinate for each frame, t and t', respectively for the K and K' frame. We synchronize the clocks so that t = t' = 0 when the origins overlap.

The common-sense classical relationship between the coordinates (x,t) in the K frame can readily be round since for any specific point x' we have x = vt + x'. We obtain

$$x' = x - vt \quad \text{and} \quad t' = t \text{ (absolute time for everyone).}$$

The second equation means absolute time. The two clocks run at the same rate. This is known as the Galilean transformation. However, this transformation is valid only for small speeds v. When the relative speed between the frames is great, i.e., appreciable compared to the speed of light, it is the Lorentz transformation that is valid.

B4. The Lorentz Transformation



Hendrik Antoon Lorentz (1853-1928)
Courtesy School of Mathematics and Statistics
University of St. Andrews, Scotland

The Lorentz transformation between the frames K and K':

$$x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}} \quad \text{and} \quad t' = \frac{t - \frac{vx}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}} .$$

Lorentz arrived at this to explain the Michelson-Morley experiment (1887), which indicated light appeared to travel at the same speed independent of the observer. The Dutch physicist Lorentz by the way shared the Nobel Prize in Physics in 1902 with Zeeman for the discovery and explanation of the Zeeman effect.

B5. Special Relativity

Albert Einstein (1879-1955)
Courtesy School of Mathematics and Statistics
University of St. Andrews, Scotland

Albert Einstein put forth the Theory of Special Relativity in 1905. The postulates are:

1. First Postulate. The Laws of Physics are the same in all inertial frames.

2. Second Postulate: The speed of light (in vacuum) is the same in all inertial frames.

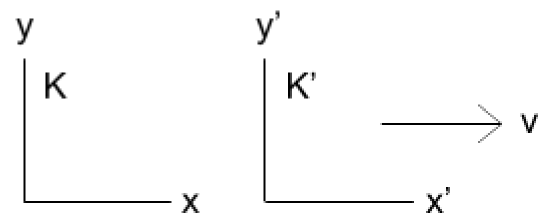
From the second postulate we can derive the Lorentz transformation. Einstein also published in 1905 his famous paper on the Photoelectric Effect, which won him the 1921 Nobel Prize in Physics. And this is not all he published that year.

We will derive the Lorentz transformation in a Feynmanesque manner, i.e., in a very elegant way.

Consider a light beam emitted at the origin when the clocks are synchronized at t = t'. For the two frames we have

$$x = ct \text{ and } x' = ct',$$

where the speed of light is taken to be the same according to Einstein's Second Postulate.



To allow for light traveling in the positive x and negative x directions we square both sides of each equation, arriving at

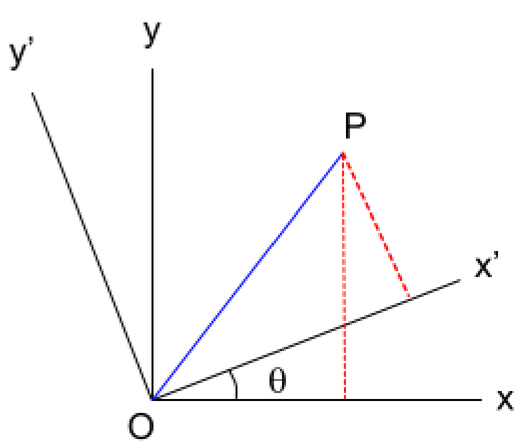
$$x^2 = c^2t^2 \text{ and } (x')^2 = c^2(t')^2 .$$

We can then state the following elegant result.

$$x^2 - c^2t^2 = (x')^2 - c^2(t')^2$$

This arrangement is the same in each frame. We say that the quantity $x^2 - c^2t^2$ is invariant. Einstein was always looking for that which does not change from frame to frame such as the speed of light c and relations like this one.

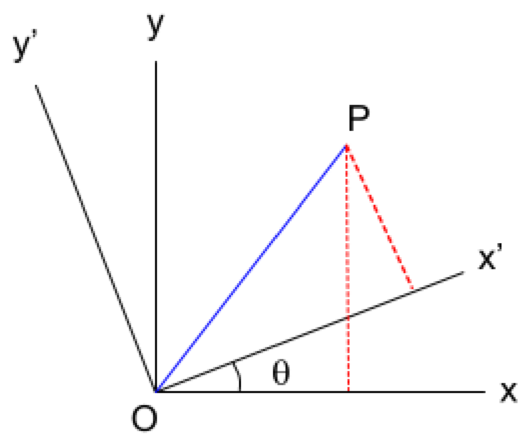
What is invariant in our rotation scheme below? The length of the OP line. Coordinates may vary but each frame agrees on the distance between O and P.



The mathematician Hermann Minkowski introduced the trick of letting $y = ict$. Then, the invariant is

$$x^2 + y^2 = x^2 - c^2t^2 .$$

WARNING: This y is NOT our spatial y dimension in the reference frame above. The y = ict is related to our time variable. We still have our regular y in our room.



Our coordinate transformation

$$x' = x\cos\theta + y\sin\theta$$

$$y' = -x\sin\theta + y\cos\theta$$

with the Minkowski trick: $y = ict$ and $y' = ict'$, gives us the Lorentz transformation - well, almost. We still have to deal with the angle.

$$x' = x\cos\theta + ict\sin\theta$$

$$ict' = -x\sin\theta + ict\cos\theta$$

We have an analogy here. The rotations with angles are analogous to our reference frames with various speeds v. All we have to do is determine how the angle theta relates to the speed. We do this by looking at a point that stays put in the K' frame so that $\Delta x' = 0$.

Then, the observed change in that position measured by the K frame is $v = \dfrac{\Delta x}{\Delta t}$ as the K frame watches the K' frame moving on. We can write

$$\Delta x' = \Delta x\cos\theta + ic\Delta t\sin\theta = 0$$

$$\Delta x\cos\theta = -ic\Delta t\sin\theta$$

$$\frac{\Delta x}{\Delta t} = -ic\frac{\sin\theta}{\cos\theta}$$

$$v = -ic\tan\theta$$

$$\tan\theta = -\frac{v}{ic} = i\frac{v}{c}$$

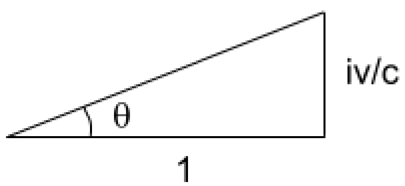
Now this would freak out most folks. The tangent is a slope. You can't have an imaginary slope. Thus we meet a characteristic of Feynman magic with math. Such an imaginary slope wood not phase Feynman in the least and he would continue on as follows. In summary, we have so far

$$x' = x\cos\theta + ict\sin\theta$$

$$ict' = -x\sin\theta + ict\cos\theta$$

$$\tan\theta = i\frac{v}{c}$$

Proceeding as if nothing is strange, we set up our right triangle so that the tangent of the angle is correct. We then determine the hypotenuse with Pythagorean's Theorem, not phased in the least with the imaginary number.



$$hypotenuse = \sqrt{1^2 + \left[i\frac{v}{c}\right]^2} = \sqrt{1 - \frac{v^2}{c^2}}$$

This is amazing. Filling in for the trig we have

$$x' = x\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} + ict\frac{i\frac{v}{c}}{\sqrt{1-\frac{v^2}{c^2}}}$$, which gives $$x' = \frac{x - vt}{\sqrt{1-\frac{v^2}{c^2}}}$$;

$$ict' = -x\frac{i\frac{v}{c}}{\sqrt{1-\frac{v^2}{c^2}}} + ict\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$, which gives $$t' = \frac{t - x\frac{v}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}$$.

I first saw this cool derivation in L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields* (Pergamum Press, Oxford, 1962).