

## Electromagnetic Theory

Prof. Ruiz, UNC Asheville, doctorphys on YouTube

### Chapter G Notes. Maxwell's Equations: Integral Form

#### G1. No Magnetic Monopoles

$$\oiint \vec{E} \cdot d\vec{A} = \frac{Q}{\epsilon_0}$$

$$\oiint \vec{B} \cdot d\vec{A} = 0$$

$$\oint \vec{E} \cdot d\vec{l} = -\frac{d\Phi_B}{dt}$$

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I$$

The equations at the left summarize what we have so far. Below are the steps we took.

1. We started with Coulomb's Law. The first equation is Gauss's Law, which is an alternate form of Coulomb's Law.

2. Then we used special relativity and Coulomb's Law to arrive at the magnetic field. The field lines closed on themselves for our line of current. Therefore, there are no net magnetic-field piercings through an enclosed surface. In other words, what goes in must come out. We will come back to this after we review the last two equations.

3. The third equation is Faraday's Law, which we derived using our knowledge of electric and magnetic fields as well as the Lorentz force law.

4. The last equation is Ampère's Law, which ultimately came from our derivation of the magnetic field from Coulomb's Law and special relativity.

Now back to monopoles. Current through a wire and magnets both set up magnetic fields. Then, when a nearby charge moves, there is a sideways force to or away from the current or the magnet.

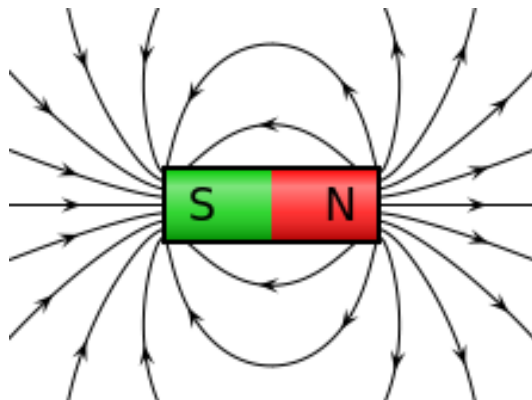


Figure Courtesy Geek3, Wikimedia Commons

If you sketch any enclosed volume, you have no net magnetic field lines leaving or entering the volume. What goes in comes out and vice versa.

If your volume region swallows up the entire magnet, north and south cancels. There is no net pole. This would be like enclosing a plus and negative charge. The total charge inside would be zero.

Now, if one north or south pole existed alone, that would be different. But no one has ever found one pole by itself, the so called monopole. Some believe these should exist. If so, then your B-field surface integral equation would look like Gauss's Law.

## G2. The Displacement Current



**James Clerk Maxwell (1831-1879)**

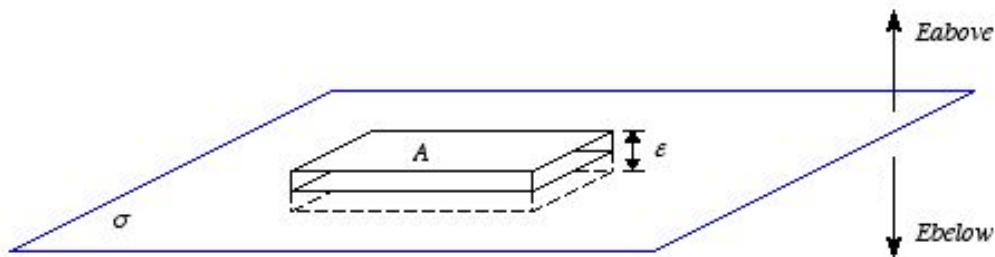
Courtesy School of Mathematics and Statistics  
University of St. Andrews, Scotland

Maxwell focused on the fact that a changing magnetic flux produces an electric field. For cases when the magnetic field strength increases or decreases through a given area one can say that a changing magnetic field produces an electric field.

Could the reverse be true? Could a changing electric field produce a magnetic field? In other words, could a changing electric flux mean we get a magnetic field?

Maxwell added the piece that a changing electric field indeed produces a magnetic field, which makes possible electromagnetic waves.

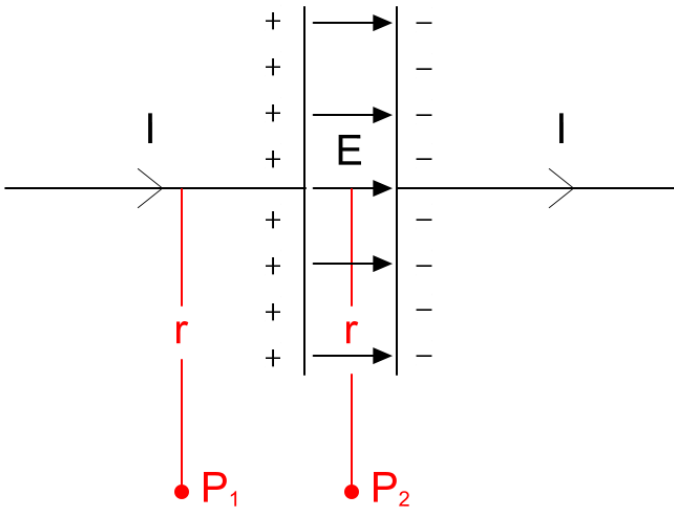
Before we analyze Maxwell's hunch, we will need a calculation you did in your intro physics course with Gauss's Law. To be complete, we will repeat it here. The problem is to find the electric field due to an infinite sheet of charge with density  $\sigma$  per unit area. We make a little rectangle box to enclose a piece of the plane inside.



Courtesy Prof. Frank L. H. Wolfs, Department of Physics  
and Astronomy, University of Rochester, NY

We apply Gauss's Law  $\oiint \vec{E} \cdot d\vec{A} = \frac{Q}{\epsilon_0}$ , where  $Q$  is the charge inside. The electric field is upward on the above surface and downward on the below surface. The result is

$$EA + EA = \frac{\sigma A}{\epsilon_0}, \text{ which gives } E = \frac{\sigma}{2\epsilon_0}.$$



Earlier we worked out the magnetic field a distance  $r$  (see point  $P_1$ ) from a wire with current  $I$  using Ampère's law

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I$$

and found  $B = \frac{\mu_0 I}{2\pi r}$ .

What happens if we interrupt the current by placing two finite plates each with area  $A$  in the path? There

is no current across the gap. Do we get  $B = 0$  outside the plate region at say point  $P_2$ ? The two parallel plates form a capacitor, a circuit element that stores charge. Those plates are charging up and the plates stop the current from going across the gap.

We will approximate the electric field inside as the sum of two "large" sheets of charge,

neglecting edge effects. Since each sheet produces  $E = \frac{\sigma}{2\epsilon_0}$  and the opposite

charges on each side work together to produce an even stronger electric field, the total strength due to both sheets is  $E = \frac{\sigma}{\epsilon_0}$ , i.e., double. The charge density is  $\sigma = \frac{Q}{A}$ ,

where  $Q$  is the absolute magnitude of the total charge on each plate and the area of each plate is  $A$ .

In the spirit of Maxwell's insight, we find the change in electric flux between the plates.

$$\frac{d\Phi_E}{dt} = \frac{d(EA)}{dt} = \frac{d\left[\frac{\sigma}{\epsilon_0} A\right]}{dt} = \frac{d\left[\frac{Q}{\epsilon_0}\right]}{dt} = \frac{I}{\epsilon_0}.$$

Remember that there is no actual current  $I$  between the plates. But there is indeed an

increasing electric flux  $\frac{d\Phi_E}{dt}$ , which can be expressed in terms of current elsewhere.

That's what the above equation is telling us.

We make the bold statement that the magnetic field  $\vec{B}$  should be the same outside at point  $P_2$  due to this effect too. Then we need  $\oint \vec{B} \cdot d\vec{l} = \mu_0 I$  to apply at point  $P_2$  also, even though there is no actual current  $I$  if you make the perpendicular trip from  $P_2$  to the axis along which the wire is located. Instead we have  $\frac{d\Phi_E}{dt} = \frac{I}{\epsilon_0}$ . So we need

$\mu_0 \epsilon_0 \frac{d\Phi_E}{dt} = \mu_0 \epsilon_0 \frac{I}{\epsilon_0} = \mu_0 I$  to make it work, i.e., to get the same magnetic field at point  $P_2$  as we find at point  $P_1$ . The result is to add this piece to Ampère's Law:

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I + \mu_0 \epsilon_0 \frac{d\Phi_E}{dt}.$$

The added term is called the displacement current due to its ability to displace charges in a solid. We can think of it as a "magical displacement" of current across our gap. The four equations are now complete and named after Maxwell for adding this crucial term.

### The Maxwell Equations and the Lorentz Force Law

$$\oiint \vec{E} \cdot d\vec{A} = \frac{Q}{\epsilon_0}$$

$$\oiint \vec{B} \cdot d\vec{A} = 0$$

$$\oint \vec{E} \cdot d\vec{l} = -\frac{d\Phi_B}{dt}$$

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I + \mu_0 \epsilon_0 \frac{d\Phi_E}{dt}$$

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$

The stage is set for light since now we have two laws: a changing  $B$  field creates an  $E$  field and vice versa, i.e., a changing  $E$  field creates a  $B$  field. The details will be worked out later. But now let's investigate this symmetry in the next section.

### G3. Free-Space Symmetry in the Maxwell Equations

In this section we look at the Maxwell equations in free space. There are no charges and no currents. The right column below shows the Maxwell equations with no sources of charge and no sources of current.

Maxwell Equations

$$\oiint \vec{E} \cdot d\vec{A} = \frac{Q}{\epsilon_0}$$

$$\oiint \vec{B} \cdot d\vec{A} = 0$$

$$\oint \vec{E} \cdot d\vec{l} = -\frac{d\Phi_B}{dt}$$

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I + \mu_0 \epsilon_0 \frac{d\Phi_E}{dt}$$

Maxwell Equations in Free Space

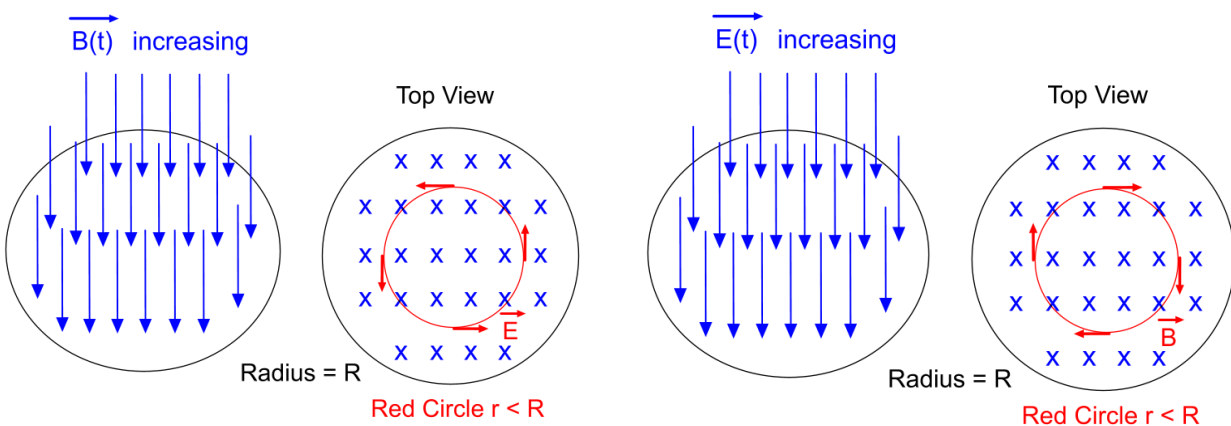
$$\oiint \vec{E} \cdot d\vec{A} = 0 \quad (\text{no charges})$$

$$\oiint \vec{B} \cdot d\vec{A} = 0$$

$$\oint \vec{E} \cdot d\vec{l} = -\frac{d\Phi_B}{dt}$$

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 \epsilon_0 \frac{d\Phi_E}{dt} \quad (\text{no currents})$$

We have already seen that a region of increasing magnetic flux through a circular region with radius  $r$  produces electric field lines tangent to the perimeter circle. See the red circle below in the left figure.



Increasing electric flux gives opposite tangent results due to the different signs below.

$$\oint \vec{E} \cdot d\vec{l} = -\frac{d\Phi_B}{dt} \quad \oint \vec{B} \cdot d\vec{l} = \mu_0 \epsilon_0 \frac{d\Phi_E}{dt}$$

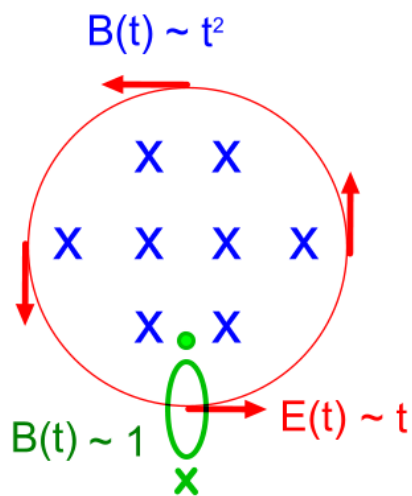
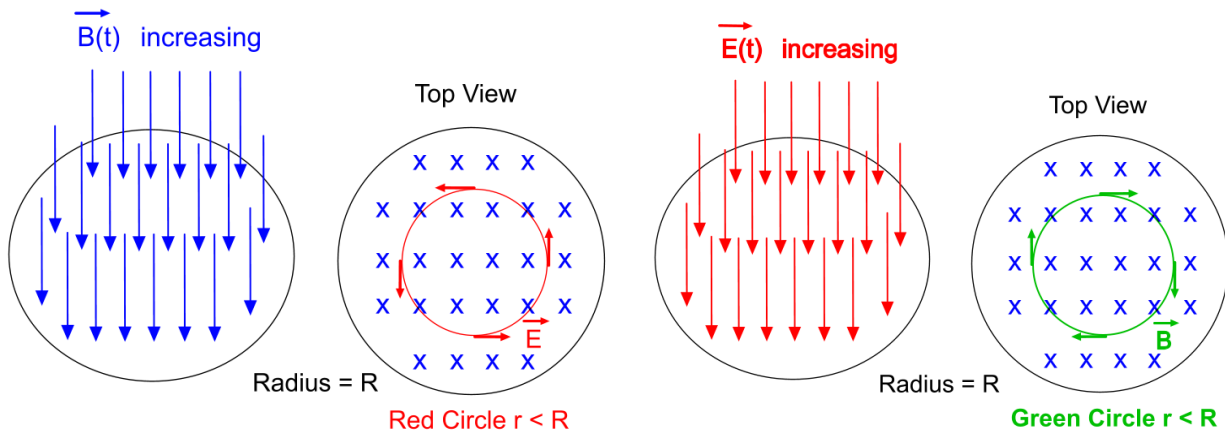
We are going to apply these rules in three dimensions. It is helpful for us to have a quick way to get the directions correct. Since Lenz's Law has the negative sign we will use our left hand to get the direction of the E field quickly. **Remember Left for Lenz.**

Use Left Hand (Thumb pointing along increasing B. Fingers curl with generated E field).

Use Right Hand (Thumb pointing along increasing E. Fingers curl with generated B field).

Mnemonic: **B**egin **L**eft  
(as in reading a line of text)  
**B** field, use **L**eft hand.

Mnemonic: **E**at **R**ight  
(as in health and nutrition)  
**E** field, use **R**ight hand.



We would like to verify here that nature always opposes us. The figure at the left indicates an initial  $B(t)$  varying as a quadratic function of time  $t$ . This increasing magnetic flux produces the circular  $E$  vector field in the counterclockwise direction using the B-Left right.

The produced  $E$  field is linear in time  $t$ . Using the E-right rule we sketch out in green the constant  $B$  field it produces. Note that on the inside of the loop that this  $B$  field opposes the original one. All is well!

If the original  $B$  field is some sinusoidal function, then you can keep taking derivatives and get a chain of  $E, B, E, B,$  etc. fields. We are one step closer to understanding the emergence of electromagnetic (EM) waves. We will convincingly demonstrate EM waves when we solve the Maxwell equations in free space and obtain the wave equation. And the speed will be, as we have noted earlier,

$$c = \frac{1}{\sqrt{\epsilon_0 \mu_0}}$$