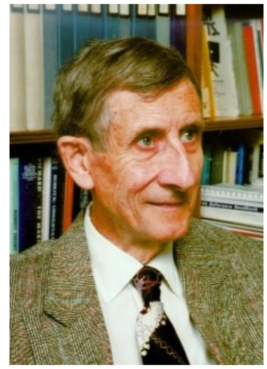
Electromagnetic Theory
Prof. Ruiz, UNC Asheville, doctorphys on YouTube
Chapter W Notes. Dirac Delta Function

W1. The 1/r Problem in Electromagnetic Theory

Sin-Itiro Tomonaga Julian Schwinger Richard P. Feynman

The Nobel Prize in Physics 1965 was awarded jointly to Sin-Itiro Tomonaga, Julian Schwinger and Richard P. Feynman *"for their fundamental work in quantum electrodynamics, with deep-ploughing consequences for the physics of elementary particles"*.



Freeman Dyson

Left Figure from Nobelprize.org and right figure from IAS School of Natural Sciences

The infinity at r = 0 for the potential energy of a point charge is a cornerstone problem that led to the development of quantum electrodynamics. Three theoretical physicists independently worked out how to handle scattering problems in electromagnetic theory, where their results enhanced the marriage between relativity and quantum mechanics initiated by Dirac. Their achievement is called QED for quantum electrodynamics.

Tomonaga's method was closely aligned with Schwinger's, both using advanced mathematical constructs. Meanwhile, Feynman got the same results writing down calculational components keyed to diagrams, the famous Feynman diagrams. Later, Freeman Dyson proved that the Feynman approach and the Schwinger/Tomonaga method were equivalent formulations using different mathematics.

In this chapter we study the 1/r problem and introduce the Dirac delta function to get a handle on things. This is only one small step in the direction towards deeper and deeper understanding of the electron's self energy. To understand QED on takes Modern Physics, Quantum I, Quantum II and then goes off to grad school, where Graduate Quantum I and Graduate Quantum II is taken - then Relativistic Quantum Mechanics I and II. We are talking 7 courses here in quantum mechanics!

W2. The Point Charge. Let's look at the simplest charge distribution, a point charge Q. It has some subtleties at r = 0 due to the $1/r^2$.

$$\vec{E} = \frac{kQ}{r^2}\hat{r} \qquad V(r) = \frac{kQ}{r}$$

Let's check this with the gradient in spherical coordinates:

$$\vec{E} = -\nabla V$$

$$\nabla V = \frac{\partial V}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\hat{\phi} .$$

Due to the spherical symmetry, $V = V(r)$. Therefore, we only have the first part of the above gradient:

$$\vec{E} = -\nabla V = -\frac{dV}{dr}\hat{r}$$

$$\vec{E} = -\frac{d}{dr}(\frac{kQ}{r})\hat{r} = -kQ\frac{d}{dr}(\frac{1}{r})\hat{r}$$

$$\vec{E} = \frac{kQ}{r^2}\hat{r}$$

The Maxwell equation $\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$ leads toes Poisson's equation: $\nabla^2 V = -\frac{\rho}{\varepsilon_0}$.

If we are not at the charge itself, then we have Laplace's equation: $\nabla^2 V = 0$. Let's check this in spherical coordinates.

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial}{\partial r}) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\frac{\partial}{\partial \theta}) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial \phi^2}$$

Due to the spherical symmetry we only have the radial part:

$$\nabla^2 V(r) = \frac{1}{r^2}\frac{d}{dr}\left[r^2 \frac{dV}{dr}\right]$$, where $V(r) = \frac{kQ}{r}$ and $k = \frac{1}{4\pi\varepsilon_0}$.

$$\nabla^2 V(r) = \frac{1}{r^2}\frac{d}{dr}\left[r^2 \frac{d}{dr}(\frac{kQ}{r})\right]$$

$$\nabla^2 V(r) = \frac{kQ}{r^2}\frac{d}{dr}\left[r^2 \frac{d}{dr}\frac{1}{r}\right]$$

$$\nabla^2 V(r) = \frac{kQ}{r^2}\frac{d}{dr}\left[-r^2 \frac{1}{r^2}\right]$$

$$\nabla^2 V(r) = -\frac{kQ}{r^2}\frac{d}{dr}[1]$$

Okay, this is indeed zero outside the charge, i.e., for $r > 0$. But what about at $r = 0$? But there is a 1/r² and 1/r in

$$\nabla^2 V(r) = -\frac{kQ}{r^2}\frac{d}{dr}\left[r^2 \frac{d}{dr}\frac{1}{r}\right].$$

Watch what happens if we integrate over a small volume surrounding the charge, using τ for volume since V stands for potential. We will use the divergence theorem.

$$\iiint_\tau \nabla^2 V(r) d\tau = \iiint_\tau \nabla \cdot \nabla V(r) d\tau = \oiint \nabla V(r) \cdot \overrightarrow{da}$$

$$= \oiint \nabla\left[\frac{kQ}{r}\right] \cdot \overrightarrow{da} = -\oiint \frac{kQ}{r^2}\hat{r} \cdot \overrightarrow{da} = -\frac{kQ}{r^2}4\pi r^2 = -4\pi kQ$$

We can summary this result with

$$\iiint_\tau \nabla^2 V(r)\,d\tau = -4\pi k Q = -\frac{Q}{\varepsilon_0}.$$

This integrated out fine. But a strange feature is that this result is independent of the volume we choose. The $1/r^2$ cancels the r^2 in the area part. Yet this finite result

$$-\frac{Q}{\varepsilon_0} = \frac{-\int \rho\, d\tau}{\varepsilon_0}$$

is coming solely from the point charge since $\rho = 0$ everywhere except for $r = 0$.

This means $\nabla^2 V = -\frac{\rho}{\varepsilon_0}$ gives zero everywhere except for $r = 0$. We have shown $\nabla^2 V = 0$ for $r > 0$, but do not know what to do write for $r = 0$. So we are going to make up a notation that captures the physics. We will write

$$\nabla^2 V = -\frac{Q}{\varepsilon_0}\delta(\vec{r})\text{, where}$$

1) $\delta(\vec{r}) = 0$ for $r > 0$ so that we get $\nabla^2 V = 0$ for $r > 0$ as expected,

2) $\iiint_\tau \nabla^2 V(r)\,d\tau = -\frac{1}{\varepsilon_0}\iiint_\tau Q\delta(\vec{r})\,d\tau \equiv -\frac{Q}{\varepsilon_0}$ as expected.

This forces us to $\rho(r) = Q\delta(\vec{r})$ and $\iiint_\tau \delta(\vec{r})\,d\tau = 1$.

The function $\delta(\vec{r})$ is called the Dirac delta function. In three dimensions the units are inverse volume so that $\rho(r) = Q\delta(\vec{r})$ has dimensions charge per volume. The one-dimensional version has dimensions inverse length and is

$$\delta(x) = 0 \text{ for } x > 0 \text{ and } \int_{-\infty}^{+\infty} \delta(x)\,dx = 1.$$

So far everything looks sound. **Now we will make mathematicians very nervous!**

W3. The Dirac Delta Function

Paul Dirac (1902-1984)

Courtesy School of Mathematics and Statistics
University of St. Andrews, Scotland

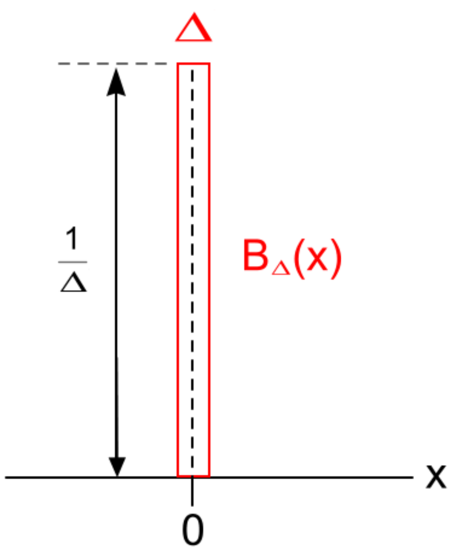
Consider the following crazy function:

$$\delta(x) = \begin{cases} 0, & x \neq 0 \\ +\infty, & x = 0 \end{cases}$$

with

$$\int_{-\infty}^{\infty} \delta(x)\, dx = 1 .$$

No, no, no! You can't do this. How can you even calculate an area where there is no thickness? First, the box "function."



The Box Function B(x). The Box is centered over x = 0.

$$B_\Delta(x) = 0 \text{ outside the } \Delta \text{ region}$$

$$B_\Delta(x) = \frac{1}{\Delta} \text{ in the } \Delta \text{ region}$$

The area for this function is

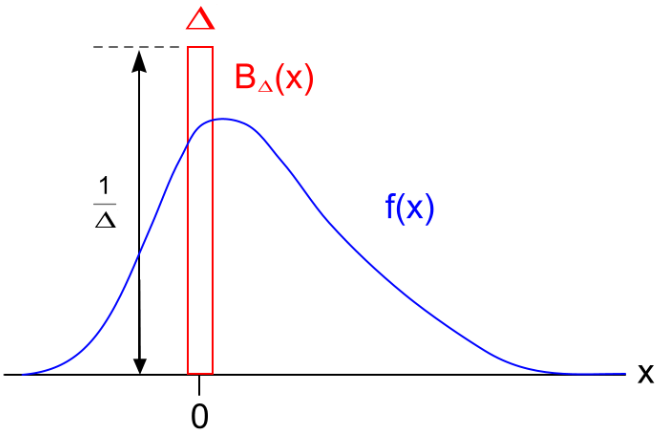
$$\int_{-\infty}^{\infty} B_\Delta(x) dx = \frac{1}{\Delta}\Delta = 1$$

The family of box functions all have area 1. As Δ gets smaller and smaller, the box gets taller and thinner. Then,

$$\lim_{\Delta \to 0} B_\Delta(x) = \delta(x) .$$

If we replace the box function with a series of Gaussians that get taller and thinner, then mathematicians are happy with the delta function defined as the limit of narrowly-peaked continuous functions. We will return to the concept of narrowly-peaked continuous functions after we consider the sifting property.

The Sifting Property. We integrate the box function with an arbitrary function f(x). Then, by the **mean value theorem** for integration,



$$\int_{-\infty}^{\infty} B_\Delta(x) f(x) dx \overset{MVT}{=} \frac{1}{\Delta} f(c)\Delta,$$

where x = c is somewhere in the Δ region. The mean value theorem for integrals states that we can find some "average" height f(c) in the Δ region so that f(c) Δ gives the area of the product function for our strip. Therefore,

$$\int_{-\infty}^{\infty} B_\Delta(x) f(x) dx \overset{MVT}{=} f(c)$$

If we make the box thinner and thinner, we squeeze "c" to be zero.

$$\lim_{\Delta \to 0} \int_{-\infty}^{\infty} B_\Delta(x) f(x) dx \overset{MVT}{=} \lim_{\Delta \to 0} f(c) = f(0)$$

For the super tall, super thin box, we have our delta function.

$$\lim_{\Delta \to 0} \int_{-\infty}^{\infty} B_\Delta(x) f(x) dx = \int_{-\infty}^{\infty} \delta(x) f(x) dx$$

So we write the shifting result as

$$\int_{-\infty}^{\infty} \delta(x) f(x) dx = f(0).$$

PW1 (Practice Problem). Show that

$$\int_{-\infty}^{\infty} \delta(x-a) f(x) dx = f(a).$$

Physicists are happy. **But now we will make mathematicians happy too!**

W4. A Delta Sequence of Gaussians

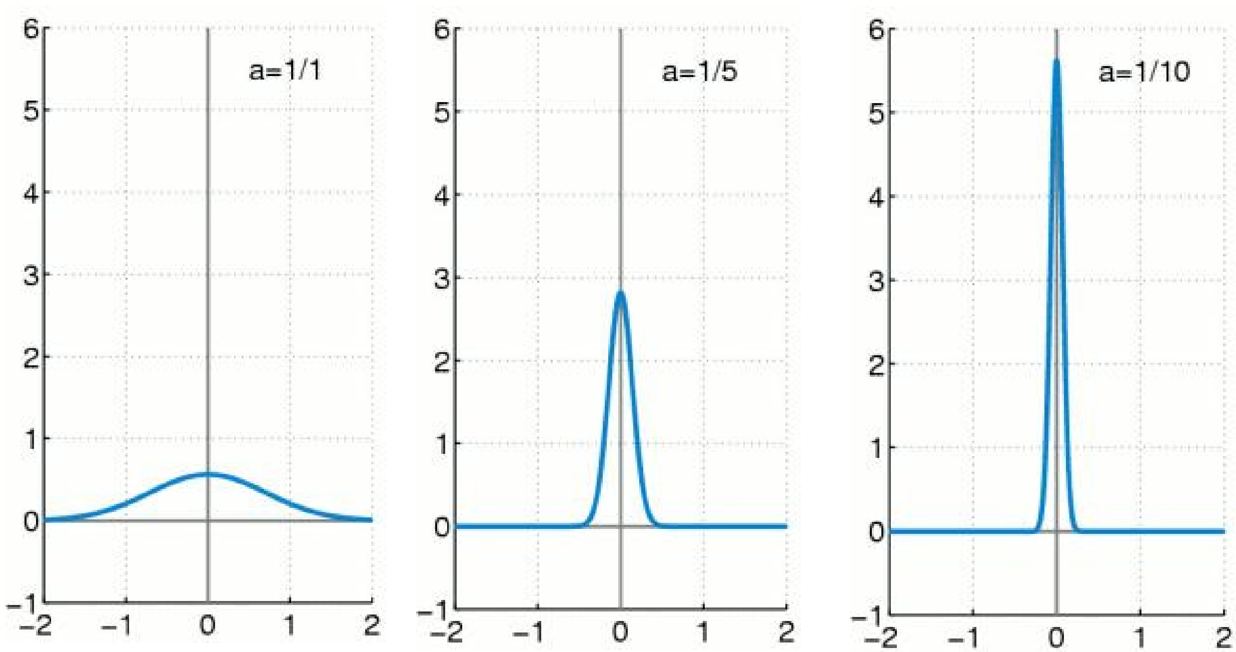
Now it is time to clean things up related to the delta function and mathematicians will calm down. We can arrive at a mathematically sound approach to the delta function by considering the delta function as a limit of a sequence of Gaussians. The Gaussian probability function is below. This function integrated over all x gives us 1.

$$P(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}}$$

Here is a sequence example from Wikipedia. Let $a^2 = 2\sigma^2$. Then define

$$\delta_a(x) = \frac{1}{a\sqrt{\pi}} e^{-\frac{x^2}{a^2}} .$$

Check out the graphs for three values for *a* as *a* gets smaller and smaller.



Courtesy Wikipedia

So we write

$$\delta(x) = \lim_{a \to 0} \delta_a(x) = \lim_{a \to 0} \left[\frac{1}{a\sqrt{\pi}} e^{-\frac{x^2}{a^2}} \right]$$

or

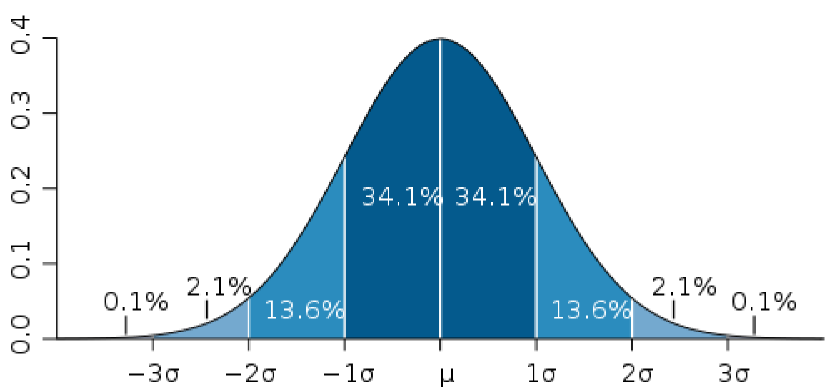
$$\delta(x) = \lim_{\sigma \to 0} \delta_\sigma(x) = \lim_{\sigma \to 0} \left[\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}} \right]$$

The standard deviation σ heads to zero for the no-spread super-tall Dirac delta function.

The General Gaussian in Statistics

$$P(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

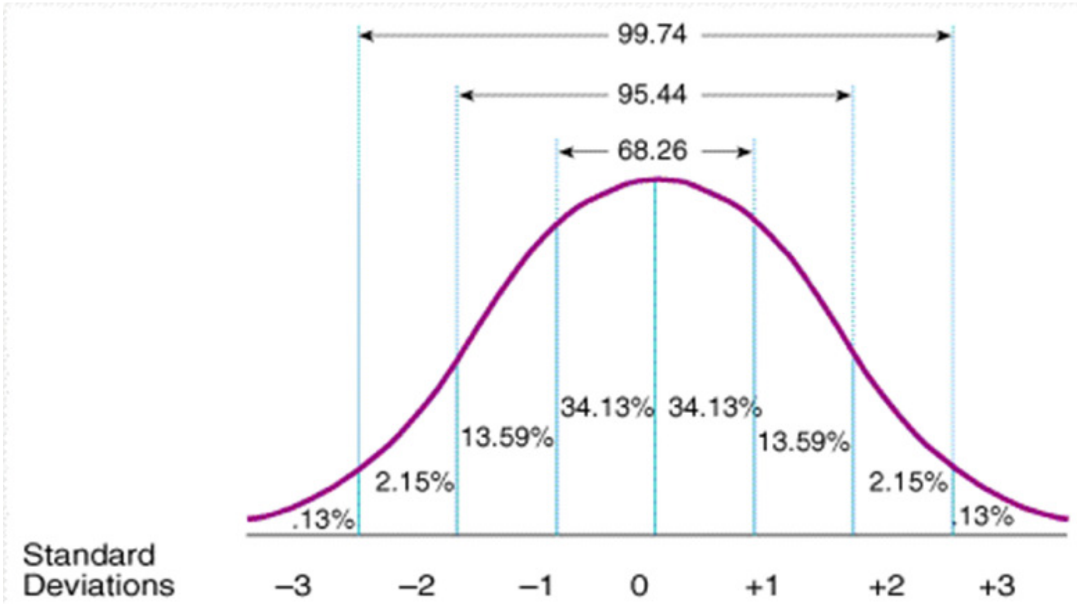
Gaussian Courtesy Petter Strandmark, Wikimedia



Measurements of any kind give Gaussians as the number of measurements gets large. The most likely value is μ and σ is called the standard deviation. Note the percentages for the area strips away from the center by σ, 2σ, 3σ.

In social science classes, it is practical to memorize these: about 68% for within plus or minus one standard deviation, 95% for within plus or minus 2σ and over 99% for within plus or minus 3σ. When you are more precise, using more significant figures and rounding off last, you have the results below.

Some remember this as the 68-95-99.7 rule.



Courtesy Hofstra