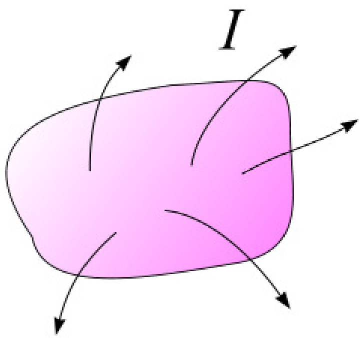
Electromagnetic Theory
Prof. Ruiz, UNC Asheville, doctorphys on YouTube
Chapter Y Notes. Electromagnetic Field Tensor

Y1. Continuity Equation. We can relate the charge density to the current density using a conservation argument. Consider a case where charge is leaving a volume as current flows outward. The current flowing out is



$$I = \oint\!\!\oint \vec{J} \cdot \vec{da} \, .$$

Since charge is leaving the region to flow out

$$I = -\frac{dq_{inside}}{dt} \, .$$

But $q_{inside} = \iiint_\tau \rho d\tau$.

Therefore, $I = \oint\!\!\oint \vec{J} \cdot \vec{da} = -\frac{\partial}{\partial t} \iiint_\tau \rho d\tau$,

where we use the partial derivative since $\rho = \rho(x, y, z, t)$.

$$\oint\!\!\oint \vec{J} \cdot \vec{da} = -\iiint_\tau \frac{\partial \rho}{\partial t} d\tau$$

Now we use the divergence theorem on the left side to obtain a volume integral.

$$\iiint_\tau \nabla \cdot \vec{J} d\tau = -\iiint_\tau \frac{\partial \rho}{\partial t} d\tau$$

$$\iiint_\tau (\nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t}) d\tau = 0$$

Now invoke the arbitrary volume rule to arrive at the beautiful continuity equation.

$$\boxed{\nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0}$$

Y2. Wizardry with the Maxwell Equations. There are four Maxwell equations that form the basis for electromagnetic theory.

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times B = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

Watch the magic as we take away our second Maxwell equation and essentially lose nothing!

Take the last equation

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

Take the divergence of a curl and you get zero.

$$\nabla \cdot (\nabla \times \vec{E}) = 0$$

We proved this already - a one-liner using Einstein's summation convention. Here it is again.

$$\nabla \cdot (\nabla \times \vec{E}) = \frac{\partial}{\partial x_l} \hat{e}_l \cdot \varepsilon_{ijk} \frac{\partial E_j}{\partial x_i} \hat{e}_k = \frac{\partial}{\partial x_l} \varepsilon_{ijk} \frac{\partial E_j}{\partial x_i} \delta_{lk} = \varepsilon_{ijk} \frac{\partial^2 E_j}{\partial x_k \partial x_i} = 0$$

The symmetric differentiation in k and i gets wasted by the antisymmetric ε_{ijk} .

Therefore $0 = \nabla \cdot (\nabla \times \vec{E}) = \nabla \cdot (-\frac{\partial \vec{B}}{\partial t})$ leads to $\frac{\partial (\nabla \cdot \vec{B})}{\partial t} = 0$.

$$\nabla \cdot \vec{B} = const$$

But from the third equation, the magnetic field is perpendicular to currents in wires. It is in the $\hat{\theta}$ direction for a wire. So $\nabla \cdot \vec{B} = 0$. I am happy to conclude from this that the constant is zero.

$$\nabla \cdot \vec{B} = 0$$

The conclusion here is that we removed one Maxwell equation, the 2nd one, and are still good with a complete theory. Now we will do what seems to be impossible. We remove another Maxwell equation. We will kick the one with $\nabla \cdot \vec{E}$ so that we kick the two with the divergences. **This is removing half the theory and still having it all!**

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times B = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

We start with

$$\nabla \times B = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} .$$

Once again, the divergence of a curl is zero.

Therefore $\nabla \cdot (\nabla \times \vec{B}) = 0$.

Taking the divergence of both sides of the third Maxwell equation gives

$$\nabla \cdot (\nabla \times \vec{B}) = \mu_0 \nabla \cdot \vec{J} + \mu_0 \varepsilon_0 \frac{\partial (\nabla \cdot \vec{E})}{\partial t} = 0 .$$

Therefore,

$$\mu_0 \nabla \cdot \vec{J} + \mu_0 \varepsilon_0 \frac{\partial (\nabla \cdot \vec{E})}{\partial t} = 0$$

$$\nabla \cdot \vec{J} + \varepsilon_0 \frac{\partial (\nabla \cdot \vec{E})}{\partial t} = 0$$

Now invoke the common-sense continuity equation: **"What goes in must come out!"**

Continuity Equation: $\nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0$.

We substitute $\nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t}$ in $\nabla \cdot \vec{J} + \varepsilon_0 \frac{\partial (\nabla \cdot \vec{E})}{\partial t} = 0$ and obtain

$-\frac{\partial \rho}{\partial t} + \varepsilon_0 \frac{\partial (\nabla \cdot \vec{E})}{\partial t} = 0$, which leads to Voilà! (vwah-LAH): $\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$.

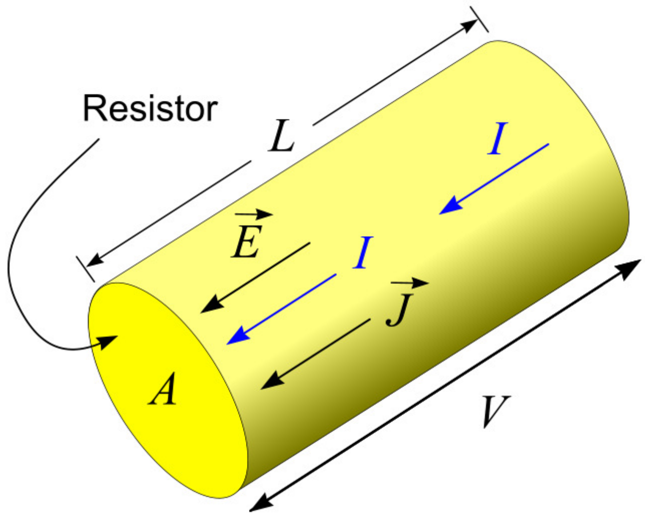
What is the analogous argument for the integration constant compared to what we used for the magnetic-field case earlier?

Y3. Resistance. The current density in a wire is proportional to the force.

$$\vec{J} = \sigma(\vec{E} + \vec{v} \times \vec{B})$$

The constant σ is called the conductivity. Velocities of charges in wire are small so we take

$$\vec{J} = \sigma\vec{E}.$$



Consider the resistor at the left.

$$J = \sigma E$$

$$\frac{I}{A} = \sigma\frac{V}{L}$$

$$\frac{IL}{\sigma A} = V$$

$$V = IR$$

$$R = \frac{L}{\sigma A}$$

The resistivity is defined as $\rho \equiv \frac{1}{\sigma}$.

So we can also write $R = \rho\frac{L}{A}$.

Our relation with V, I, and R is always true as a definition of resistance. The fact that R can often be taken to be constant is called Ohm's Law. Loosely, we say in all cases the following is Ohm's Law. When a bulb heats up, R changes, but we following still defines the resistance.

$$V = IR$$

Y4. Four Vectors

1. Spacetime. The four-vector for spacetime is

$$x^{\mu} = (ct, \vec{r})$$, called a contravariant vector.

We are using the strict notation here where a superscript is necessary for the index that takes on the four values 0, 1, 2, 3 for ct, x, y, and z respectively. The lower subscript definition is called a covariant vector and is given by

$$x_{\mu} = (ct, -\vec{r})$$ so that

$$x^{\mu} x_{\mu} = (ct)^2 - (\vec{r} \cdot \vec{r})$$, the invariant in special relativity.

The covariant four-vector can be obtained from the contravariant one with this "magic" matrix.

$$\begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix}$$

PY1 (Practice Problem). Verify that the above matrix does the trick.

We write the above matrix multiplication as $x_{\mu} = \eta_{\mu\nu} x^{\nu}$ with

$$\eta_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

The above matrix is called the metric tensor for special relativity.

2. Density Four-Vector. We suspect a four-vector for charge density ρ and current density $\vec{J}$. We will arrive at it through dimensional analysis. Remember inserting the c factor with time t so that ct has dimensions on an equal footing with x, y, and z? We want the similar deal here with our scalar variable and vector variable.

Dimensions for charge density: $[\rho] = QL^{-3}$

Dimensions for current density: $\left[\vec{J}\right] = QT^{-1}L^{-2}$

So we need to hit the ρ with LT^{-1} to pull this off. But this has velocity dimensions, which dimensions are included in the speed of light. So we want

$$J^{\mu} = (c\rho, \vec{J}).$$

The continuity equation $\nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0$ can be written as $\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{J} = 0$.

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{J} = \frac{1}{c}\frac{\partial J^0}{\partial t} + \nabla \cdot \vec{J} = \frac{\partial J^0}{\partial x^0} + \frac{\partial J^1}{\partial x^1} + \frac{\partial J^2}{\partial x^2} + \frac{\partial J^3}{\partial x^3}$$

CAUTION: The superscripts are NOT exponents!

$$\frac{\partial J^{\mu}}{\partial x^{\mu}} = 0$$

3. Potential Four-Vector. The dimensions for the scalar and vector potentials are

$$[V] = L^2T^{-2}MQ^{-1} \quad \text{and} \quad \left[\vec{A}\right] = LT^{-1}MQ^{-1}.$$

PY2 (Practice Problem). Derive these. Hint: Use the dimensions in equations

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B}), \quad \vec{E} = -\nabla V, \text{ and } \vec{B} = \nabla \times \vec{A}.$$

PY3 (Practice Problem). From PY-2 show $A^{\mu} = (\frac{V}{c}, \vec{A})$ and $A_{\mu} = (\frac{V}{c}, -\vec{A})$.

Y5. Electromagnetic Field Tensor

The relation $\vec{B} = \nabla \times \vec{A}$ leads us to the following.

$$\nabla \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \hat{i}(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}) + \hat{j}(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}) + \hat{k}(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y})$$

The pairs of partial derivatives suggest to a theoretician that we consider something like the following entity:

$$F^{\mu\nu} = \frac{\partial}{\partial x^\mu} A^\nu - \frac{\partial}{\partial x^\nu} A^\mu = \frac{\partial A^\nu}{\partial x^\mu} - \frac{\partial A^\mu}{\partial x^\nu} \quad \text{where} \quad A^\mu = (\frac{V}{c}, \vec{A}).$$

We can fill in the magnetic field components from the above cross product.

$$F^{\mu\nu} = \begin{bmatrix} F^{00} & F^{01} & F^{02} & F^{03} \\ F^{10} & F^{11} & F^{12} & F^{13} \\ F^{20} & F^{21} & F^{22} & F^{23} \\ F^{30} & F^{31} & F^{32} & F^{33} \end{bmatrix} = \begin{bmatrix} 0 & ? & ? & ? \\ ? & 0 & B_z & -B_y \\ ? & -B_z & 0 & B_x \\ ? & B_y & -B_x & 0 \end{bmatrix}$$

PY4 (Practice Problem). Show the above partial result.

Now it's time to remember the general form we found earlier: $\vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t}$.

$$E_x = -\frac{\partial V}{\partial x} - \frac{\partial A_x}{\partial t} = -c\frac{\partial A_0}{\partial x} - c\frac{\partial A_1}{c\partial t} = c\left[-\frac{\partial A^0}{\partial x^1} + \frac{\partial A^1}{\partial x^0}\right] \quad \text{since } A^1 = -A_1.$$

PY5 (Practice Problem). Derive the completed form for our matrix, which is called the electromagnetic field tensor.

$$F^{\mu\nu} = \begin{bmatrix} F^{00} & F^{01} & F^{02} & F^{03} \\ F^{10} & F^{11} & F^{12} & F^{13} \\ F^{20} & F^{21} & F^{22} & F^{23} \\ F^{30} & F^{31} & F^{32} & F^{33} \end{bmatrix} = \begin{bmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{bmatrix}$$

Gauss-Ampère Maxwell Equations. Show that the Gauss and Ampère Maxwell equations,

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \quad \text{and} \quad \nabla \times B = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

are given by

$$\frac{\partial F^{\mu\nu}}{\partial x^{\nu}} = \mu_0 J^{\mu} \text{, where } J^{\mu} = (c\rho, \vec{J}).$$

This can also be written in super short hard.

$$F^{\mu\nu}{}_{,\nu} = \mu_0 J^{\mu}$$

First let $\mu = 0$.

$$F^{\mu\nu} = \begin{bmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{bmatrix}$$

$$F^{0\nu}{}_{,\nu} = \mu_0 J^0$$

$$\frac{1}{c}\left[\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right] = \mu_0 c \rho$$

$$\nabla \cdot \vec{E} = \mu_0 c^2 \rho \qquad c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$$

$$\nabla \cdot \vec{E} = \mu_0 \frac{1}{\mu_0 \varepsilon_0} \rho$$

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$

$$F^{\mu\nu}{}_{,\nu} = \mu_0 J^{\mu} \qquad J^{\mu} = (c\rho, \vec{J})$$

Now let $\mu = 1$.

$$F^{\mu\nu} = \begin{bmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{bmatrix}$$

$$F^{1\nu}{}_{,\nu} = \mu_0 J^1$$

$$-\frac{1}{c}\frac{\partial E_x}{\partial t} + \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} = \mu_0 J_x$$

$$\nabla \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ B_x & B_y & B_z \end{vmatrix}$$

$$-\frac{1}{c}\frac{\partial E_x}{c\partial t} + \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} = \mu_0 J_x$$

$$-\frac{1}{c^2}\frac{\partial E_x}{\partial t} + (\nabla \times B)_x = \mu_0 J_x$$

$$(\nabla \times B)_x = \mu_0 J_x + \frac{1}{c^2}\frac{\partial E_x}{\partial t} \qquad c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$$

$$(\nabla \times B)_x = \mu_0 J_x + \mu_0 \varepsilon_0 \frac{\partial E_x}{\partial t}$$

The other two cases: $\mu = 2$ and $\mu = 3$ gets you the other two vector components.

$$\nabla \times B = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

The Other Two Maxwell Equations. Show that the no-monopole and Faraday Maxwell equations

$$\nabla \cdot \vec{B} = 0 \quad \text{and} \quad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

are given by

$$\frac{\partial F_{\mu\nu}}{\partial x^{\lambda}} + \frac{\partial F_{\nu\lambda}}{\partial x^{\mu}} + \frac{\partial F_{\lambda\mu}}{\partial x^{\nu}} = 0 \text{, where } F_{\mu\nu} = \eta_{\mu\alpha}\eta_{\nu\beta}F^{\alpha\beta}.$$

First we find the covariant form of the electromagnetic tensor $F_{\mu\nu}$. We proceed with one step at a time.

$$F_{\mu\beta} = \eta_{\mu\alpha}F^{\alpha\beta} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{bmatrix}$$

$$F_{\mu}{}^{\beta} = \begin{bmatrix} 0 & E_x/c & E_y/c & E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{bmatrix}$$

$$F_{\mu\nu} = \eta_{\nu\beta}F_{\mu}{}^{\beta} \qquad\qquad F_{\mu\nu} = F_{\mu}{}^{\beta}\eta_{\nu\beta} = F_{\mu}{}^{\beta}\eta_{\beta\nu}$$

$$F_{\mu\nu} = F_{\mu}{}^{\beta}\eta_{\nu\beta} = \begin{bmatrix} 0 & E_x/c & E_y/c & E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$F_{\mu\nu} = \begin{bmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & B_z & -B_y \\ E_y/c & -B_z & 0 & B_x \\ E_z/c & B_y & -B_x & 0 \end{bmatrix} = \begin{bmatrix} F_{00} & F_{01} & F_{02} & F_{03} \\ F_{10} & F_{11} & F_{12} & F_{13} \\ F_{20} & F_{21} & F_{22} & F_{23} \\ F_{30} & F_{31} & F_{32} & F_{33} \end{bmatrix}$$

$$\frac{\partial F_{\mu\nu}}{\partial x^\lambda} + \frac{\partial F_{\nu\lambda}}{\partial x^\mu} + \frac{\partial F_{\lambda\mu}}{\partial x^\nu} = 0$$

First let $\mu = 1$, $\nu = 2$, and $\lambda = 3$.

$$F_{\mu\nu} = \begin{bmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & B_z & -B_y \\ E_y/c & -B_z & 0 & B_x \\ E_z/c & B_y & -B_x & 0 \end{bmatrix}$$

$$\frac{\partial F_{12}}{\partial x^3} + \frac{\partial F_{23}}{\partial x^1} + \frac{\partial F_{31}}{\partial x^2} = 0$$

$$\frac{\partial B_z}{\partial z} + \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} = 0$$

$$\nabla \cdot \vec{B} = 0$$

Now let $\mu = 0$, $\nu = 1$, and $\lambda = 2$.

$$F_{\mu\nu} = \begin{bmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & B_z & -B_y \\ E_y/c & -B_z & 0 & B_x \\ E_z/c & B_y & -B_x & 0 \end{bmatrix}$$

$$\frac{\partial F_{\mu\nu}}{\partial x^\lambda} + \frac{\partial F_{\nu\lambda}}{\partial x^\mu} + \frac{\partial F_{\lambda\mu}}{\partial x^\nu} = 0$$

$$\frac{\partial F_{01}}{\partial x^2} + \frac{\partial F_{12}}{\partial x^0} + \frac{\partial F_{20}}{\partial x^1} = 0$$

$$-\frac{1}{c}\frac{\partial E_x}{\partial y} + \frac{\partial B_z}{c\partial t} + \frac{1}{c}\frac{\partial E_y}{\partial x} = 0$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} + \frac{\partial B_z}{\partial t} = 0$$

$$\nabla\times\vec{E}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ E_x & E_y & E_z\end{vmatrix}$$

$$\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}=-\frac{\partial B_z}{\partial t}$$

$$(\nabla\times\vec{E})_z=-\frac{\partial B_z}{\partial t}$$

The other cases: $\mu=0$, $\nu=1$, $\lambda=3$ and $\mu=0$, $\nu=2$, $\lambda=3$ give the other vector components for

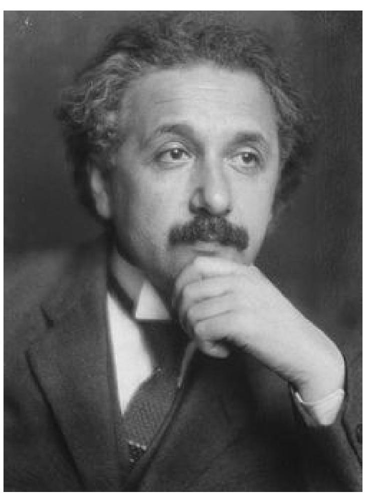
$$\nabla\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}.$$

This can also be written in super short-hard.

$$\left\{F^{\mu\nu}{}_{,\lambda}\right\}=0.$$

Summary. The Maxwell Equations - Foundation of Electromagnetic Theory.

$$F^{\mu\nu}{}_{,\nu}=\mu_0 J^{\mu} \qquad \left\{F^{\mu\nu}{}_{,\lambda}\right\}=0$$



Einstein's General Theory of Relativity - the Modern Theory of Gravitation has 10 field equations (16 reduce to 10).

$$G_{\mu\nu}=\frac{8\pi}{c^4}T_{\mu\nu}$$

$$R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R+g_{\mu\nu}\Lambda=\frac{8\pi}{c^4}T_{\mu\nu}$$

Einstein spent the last 30 years of his life trying to unify these equations! Einstein: "I want to know God's thoughts - the rest are mere details." Einstein (1879-1955).

THE END