Theoretical Physics
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Chapter W Notes. The Principle of Least Action

W1. Gravity, Time, and the Lagrangian.

Reference: Elisha Huggins, "Gravity, Time, and Lagrangians," *The Physics Teacher* **48**, pp. 512-514 (November 2010).

The Pound-Rebka experiment (publication in 1960) measured the frequency shift in light as light traveled down a seven-story shaft. If you drop a ball through a height z, that ball gains kinetic energy by way of the work done by the force of gravity.

$$W = \int_0^z F dz = \int_0^z mg dz = mg \int_0^z dz = mgz$$

This work is translated into kinetic energy. Remember when we discussed work earlier in our course? Now we use $F = ma = m\dfrac{dv}{dt}$.

$$W = \int_0^z m \frac{dv}{dt} dz = m\int_0^z \frac{dv}{dz}\frac{dz}{dt} dz = m\int_0^z v\frac{dv}{dz} dz = m\int_0^v v\, dv = \frac{1}{2}mv^2$$

Equating these we have

$$W = mgz = \frac{1}{2}mv^2$$

Near the Earth the gravitational field can be taken to be constant. The potential energy is defined as

$U = mgz$, where $z = 0$ is at the Earth's surface.

Then, if you fall drop a stone from a building and it falls to the ground, you get this energy translated into kinetic energy when frictional forces are neglected. For any given height, the total energy is the sum of the kinetic energy and potential energy.

$$E = \frac{1}{2}mv^2 + mgz$$

Comparing two different heights during a fall, you have

$$\frac{1}{2}mv_1^2 + mgz_1 = \frac{1}{2}mv_2^2 + mgz_2$$

If you drop a stone from rest at height $z_1 = z$, then $v_1 = 0$. If you let it fall to the ground $(z_2 = 0)$ where the final velocity is $v_2 = v$, you get the earlier result

$$0 + mgz = \frac{1}{2}mv^2 + 0$$

We are going to replace the kinetic energy with the energy of a photon. For a photon γ , the energy is given by $E_\gamma = hf$. The "falling" photon of course cannot speed up. Instead, the energy gained results in a higher frequency. So start by stating the conservation of energy for the photon that "falls" to the ground.

$$hf_{top} + mgz = hf_{bottom} + 0$$

Now use $E = mc^2$ to substitute for the mass $m_{top} = \dfrac{E_{top}}{c^2} = \dfrac{hf_{top}}{c^2}$.

$$hf_{top} + \frac{hf_{top}}{c^2}gz = hf_{bottom}$$

Let's solve for the bottom: $hf_{bottom} = hf_{top} + \dfrac{hf_{top}}{c^2}gz$

We bring the hf_{top} out to the left next.

$$hf_{bottom} = hf_{top}(1 + \frac{gz}{c^2}) \qquad f_{bottom} = f_{top}(1 + \frac{gz}{c^2})$$

$$f_{bottom} = f_{top}(1+\frac{gz}{c^2})$$

We now compare the periods. Remember $T = \frac{1}{f}$. Also, note that $\frac{gz}{c^2} \ll 1$.

Comparing the periods,

$$T_{bottom} = T_{top}(1+\frac{gz}{c^2})^{-1}.$$

Remember your Taylor expansion for 1 added to a small number from our first class months ago? It is $(1+\varepsilon)^n \approx 1+n\varepsilon$. Here we have $\varepsilon = \frac{gz}{c^2}$ and $n = -1$. Then

$$T_{bottom} = T_{top}(1-\frac{gz}{c^2}) \quad \text{and} \quad T_{top} = T_{bottom}(1+\frac{gz}{c^2}).$$

A nice quick rule to remember is this one: $\frac{1}{1+\varepsilon} \approx 1-\varepsilon$ for $|\varepsilon| \ll 1$.

Our top clock runs faster.

$$\Delta T_{top} = \Delta T_{bottom}(1+\frac{gz}{c^2})$$

During one second of bottom time, $\Delta T_{bottom} = 1$, the top clock advances an extra amount: $\Delta T_{extra} = \frac{gz}{c^2}$. Let's remember this rule as follows:

The gain per second by a clock at height z is $\Delta T = \frac{gz}{c^2}$.

We are shortly going to consider time dilation in special relativity. So as not to confuse that time effect with this one, we will label the clock-height effect. We choose GR for general relativity since general relativity deals with acceleration and gravitation. We restate our rule with the GR label.

The GR Rule: The gain per second by a clock at height z is $\Delta T_{GR} = \frac{gz}{c^2}$.

Now if that clock starts moving around we have the special relativistic time dilation we studied earlier. Engineers must keep track of both effects for a Global Positioning Satellite (GPS). For the special relativistic time dilation effect we have

$$T = T_0 / \sqrt{1 - \frac{v^2}{c^2}}$$

The T_0 is the proper time, i.e., the time kept by the clock in the moving frame being observed. Remember always that your proper time is the time kept by a watch in your pocket. Time slows down for you in the moving frame.

$$T_0 = T\sqrt{1 - \frac{v^2}{c^2}}$$

A Taylor expansion gives

$$T_0 = T(1 - \frac{v^2}{2c^2}) \qquad \Delta T_0 = (1 - \frac{v^2}{2c^2})\Delta T$$

This brings us to our second rule which we label SR for special relativity.

The SR Rule: The loss per second by a clock moving at v is $\Delta T_{SR} = -\frac{v^2}{2c^2}$.

Now it is time for Feynman's Game. There are two clocks on a table in a room. You take one and I take the other. We each travel with our clocks doing whatever we want but we must bring our clocks back in one hour according to the room clock. The winner is the one whose clock gains the most time.

$$\Delta T_{GR} = \frac{gz}{c^2} \qquad \Delta T_{SR} = -\frac{v^2}{2c^2}$$

Strategy 1: You should move your clock up as high as you can.
Strategy 2: It is a waste to move sideways That loses time for you.
Strategy 3: Don't speed up too much moving vertically as that loses time.

Your score is given by adding up all the effects for each $\Delta T = 1\,s$ tick of the room clock. For each tick of the room clock, a player's gain or loss is given by the sum of the two relativistic effects.

$$\Delta T_{player} = \Delta T_{GR} + \Delta T_{SR}$$

$$\Delta T_{GR} = \frac{gz}{c^2} \qquad \Delta T_{SR} = -\frac{v^2}{2c^2}$$

$$\Delta T_{score} = \sum_{n=1}^{3600}\left[\frac{gz_n}{c^2} - \frac{v_n^2}{2c^2}\right]$$

Now, we go over to an integral using our three rules 1)delta to d, 2)discrete variable to continuous, 3)summation sign to "snake."

$$\Delta T_{Score} = \sum_{n=1}^{3600}\left[\frac{gz_n}{c^2} - \frac{v_n^2}{2c^2}\right]\Delta n \qquad \Delta T_{Score} = \int_0^{1h}\left[\frac{gz(n)}{c^2} - \frac{v^2(n)}{2c^2}\right]dn$$

We can replace n with t for time.

$$\Delta T_{Score} = \int_0^{1h}\left[\frac{gz(t)}{c^2} - \frac{v^2(t)}{2c^2}\right]dt$$

We want to maximize this. If me multiply by $-mc^2$ you have the Lagrangian in the integrand and S is called the action. We want to minimize the action to win the game.

$$-mc^2\Delta T_{Score} \equiv S = \int_0^{1h}\left[\frac{1}{2}mv^2(t) - mgz(t)\right]dt$$

The action in more general terms is the integral of the difference between the kinetic energy and the potential energy. We also revert to the more common x variable.

$$S = \int_a^b\left[\frac{1}{2}mv^2(t) - V(x)\right]dt$$

The integrand is the Lagrangian.

$$L = \frac{1}{2}mv^2(t) - V(x)$$

W2. Least Action.

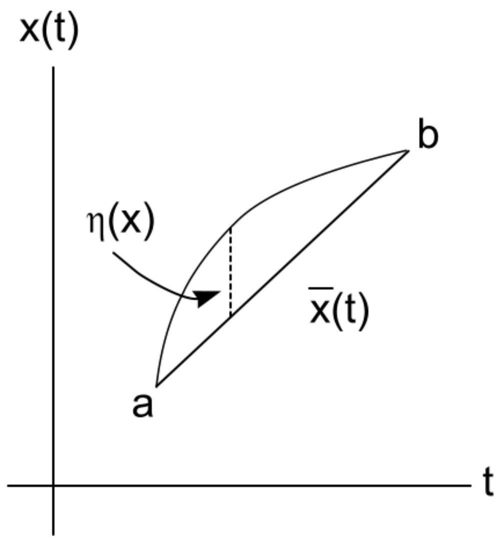
Reference: Feynman R, Leighton R, and Sands M. *The Feynman Lectures on Physics* (3 volumes) 1964, 1966. Refer to Volume 2, Chapter 19. The Principle of Least Action.

We would like to minimize the following from our game in the previous section.

$$S = \int_0^{1h} \left[\frac{1}{2} m v^2(t) - mgx(t) \right] dt$$

We see here the difference between the kinetic and potential energies. Let the integration go from a point "a" to point "b" and make the substitutions $v = \frac{dx}{dt}$ for the velocity and $V(x) = mgx$ for the potential energy. This gives us a more general expression for the potential energy. In this particular case, x represents the height.

$$S = \int_a^b \left[\frac{1}{2} m \left(\frac{dx}{dt}\right)^2 - V(x) \right] dt$$



We are in search for the ideal path that minimizes this integral. Call this ideal path $\bar{x}(t)$, i.e., the x with a bar over it. Then, an arbitrary path $x(t)$ can be expressed as a sum of the ideal path plus some deviation from the ideal.

$$x(t) = \bar{x}(t) + \eta(x)$$

For the speed we have

$$\frac{dx(t)}{dt} = \frac{d\bar{x}(t)}{dt} + \frac{d\eta(x)}{dt}$$ and for the potential $V(x) = V(\bar{x} + \eta)$.

$$S = \int_a^b \left[\frac{1}{2} m (\frac{dx}{dt})^2 - V(x) \right] dt$$

becomes

$$S = \int_a^b \left[\frac{1}{2} m (\frac{d\bar{x}}{dt} + \frac{d\eta}{dt})^2 - V(\bar{x} + \eta) \right] dt$$

For the velocity we get

$$(\frac{d\bar{x}}{dt} + \frac{d\eta}{dt})^2 = (\frac{d\bar{x}}{dt})^2 + 2\frac{d\bar{x}}{dt}\frac{d\eta}{dt} + (\frac{d\eta}{dt})^2$$

$$(\frac{d\bar{x}}{dt} + \frac{d\eta}{dt})^2 = (\frac{d\bar{x}}{dt})^2 + 2\frac{d\bar{x}}{dt}\frac{d\eta}{dt} + higher\ order\ terms$$

For the potential energy, we do a Taylor series expansion.

$$V(\bar{x} + \eta) = V(\bar{x}) + V'(\bar{x})\eta + \frac{1}{2} V''(\bar{x})\eta^2 + higher\ order$$

Note that the deviations $\eta(x)$ are small and very close to the idea path, i.e., $\eta(x) \ll 1$. So we plan to neglect higher order terms, i.e., higher powers of a small quantity.

$$S = \int_a^b \left[\frac{1}{2} m (\frac{d\bar{x}}{dt})^2 + m\frac{d\bar{x}}{dt}\frac{d\eta}{dt} - V(\bar{x}) - V'(\bar{x})\eta \right] dt$$

$$S = \int_a^b \left[\frac{1}{2} m (\frac{d\bar{x}}{dt})^2 - V(\bar{x}) + m\frac{d\bar{x}}{dt}\frac{d\eta}{dt} - V'(\bar{x})\eta \right] dt$$

The variation in the action due to the path that wanders away from the ideal is given by

$$\delta S = \int_a^b \left[m \frac{d\bar{x}}{dt} \frac{d\eta}{dt} - V'(\bar{x})\eta \right] dt$$

For the idea path, this is zero. So we set $\delta S = 0$ in search for a differential equation that will describe our ideal path.

$$\delta S = \int_a^b \left[m \frac{d\bar{x}}{dt} \frac{d\eta}{dt} - V'(\bar{x})\eta \right] dt = 0$$

The η variable is arbitrary. We want that outside the brackets but we do not have a η by itself for the other term. But we have a derivative o η. So we use integration by parts to lift the derivative off it.

$$\frac{d}{dt}\left[\frac{d\bar{x}}{dt}\eta \right] = \frac{d^2\bar{x}}{dt^2}\eta + \frac{d\bar{x}}{dt}\frac{d\eta}{dt}$$

$$\frac{d\bar{x}}{dt}\frac{d\eta}{dt} = \frac{d}{dt}\left[\frac{d\bar{x}}{dt}\eta \right] - \frac{d^2\bar{x}}{dt^2}\eta$$

$$\delta S = \int_a^b \left[m \frac{d}{dt}\left[\frac{d\bar{x}}{dt}\eta \right] - m\frac{d^2\bar{x}}{dt^2}\eta - V'(\bar{x})\eta \right] dt = 0$$

$$\delta S = m\frac{d\bar{x}}{dt}\eta\bigg|_a^b - \int_a^b \left[m\frac{d^2\bar{x}}{dt^2}\eta + V'(\bar{x})\eta \right] dt = 0$$

The integrated term $m\frac{d\bar{x}}{dt}\eta\bigg|_a^b = 0$ since you must start at the beginning point "a" and go to point "b" for any of your paths. This means there is no digression for the points at the beginning and the end. This leaves us with

$$\delta S = \int_a^b \left[m\frac{d^2\bar{x}}{dt^2}\eta + V'(\bar{x})\eta \right] dt = 0,$$

where now we can factor out the arbitrary η.

$$\delta S = \int_a^b \left[m\frac{d^2\bar{x}}{dt^2} + V'(\bar{x}) \right] \eta\, dt = 0$$

Now we use the arbitrary trick. Since the η are arbitrary deviations as we choose the different wrong paths, for the best path, everything else must be zero.

$$m\frac{d^2\bar{x}}{dt^2} + V'(\bar{x}) = 0$$

Do you recognize this? It is Newton's Second Law. Let's drop the bar now that we found our ideal path.

$$m\frac{d^2x}{dt^2} + V'(x) = 0$$

$F = ma$, where $F = -\frac{dV(x)}{dx}$ and $a = \frac{d^2x}{dt^2}$.

W3. The Lagrangian.

In our previous section we minimized the action

$$S = \int_a^b \left[\frac{1}{2} m (\frac{dx}{dt})^2 - V(x) \right] dt$$

and arrived at Newton's Second Law

$F = ma$, where $F = -\dfrac{dV(x)}{dx}$ and $a = \dfrac{d^2 x}{dt^2}$.

The integrand in the action integral is called the Lagrangian L , where

$$L = \frac{1}{2} m \left[\frac{dx}{dt} \right]^2 - V(x)$$

$$L = \frac{1}{2} m v^2 - V(x)$$

$$L = KE - PE$$

The Lagrangian is found by writing down the kinetic energy and subtracting from it the potential energy. Now we arrive at an elegant and sophisticated way to write Newton's Second Law as a minimization principle.

Start with the usual $F = ma$, where $F = -\dfrac{dV(x)}{dx}$ and $a = \dfrac{d^2 x}{dt^2}$.

$$L = \frac{1}{2} m v^2 - V(x)$$

$F = -\dfrac{dV(x)}{dx} = \dfrac{\partial L}{\partial x}$ and $ma = m\dfrac{dv}{dt} = \dfrac{d}{dt}\dfrac{\partial L}{\partial v}$

$$F = ma$$

becomes

$$\frac{\partial L}{\partial x} = \frac{d}{dt}\frac{\partial L}{\partial v}.$$

Finally, we revert to Newton's notation of a time derivative and get the following.

$$\frac{\partial L}{\partial x} = \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}$$

or

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x}$$

The above is also called an Euler-Lagrange Equation. When there are more than one dimension, you get one of these for each of the dimensions. Then they are called the Euler-Lagrange equations.