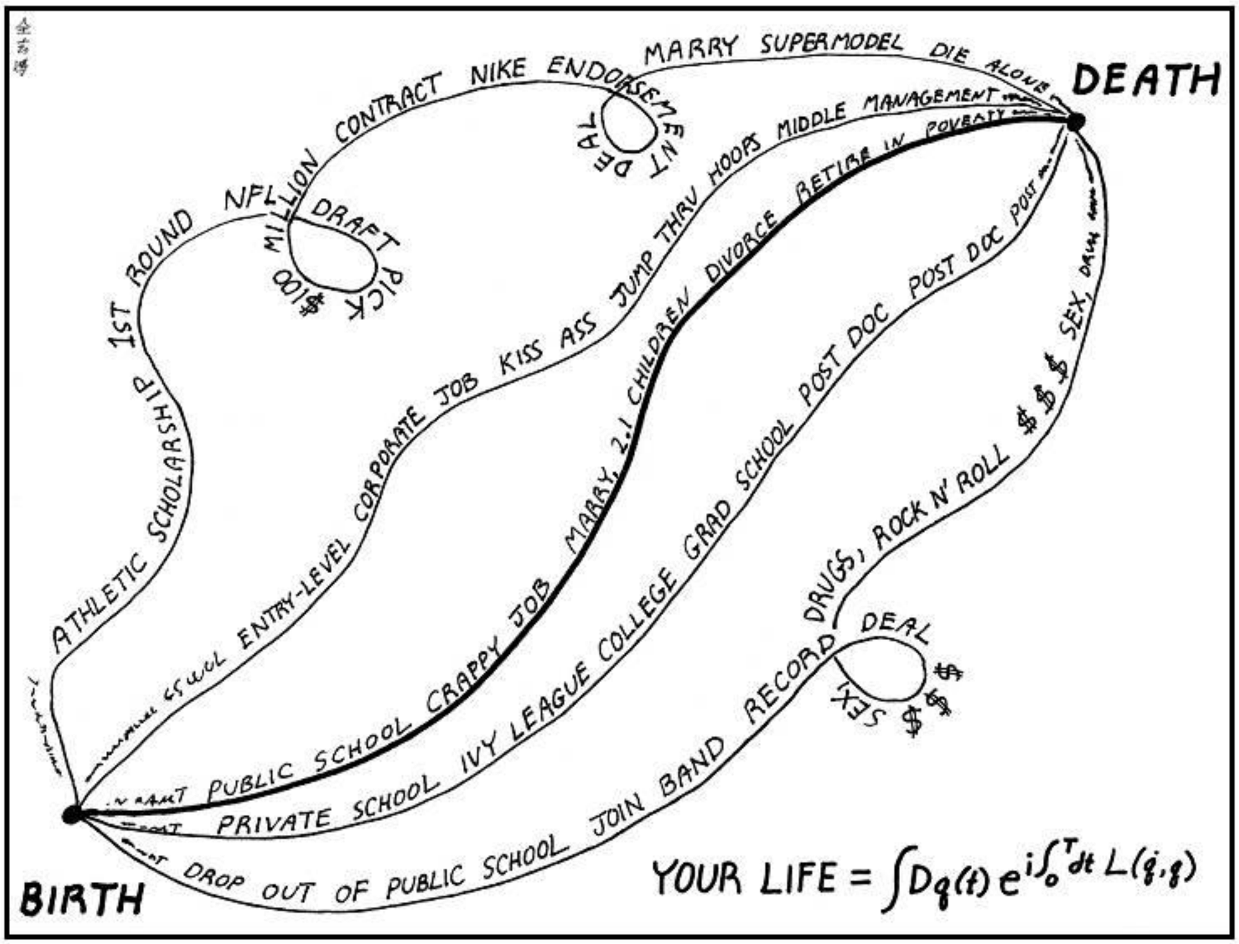
Theoretical Physics
Prof. Ruiz, UNC Asheville, doctorphys on YouTube
Chapter Y Notes. Feynman's Derivation of the Schrödinger Equation

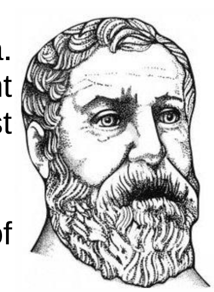
Y1. The Path Integral Approach to Quantum Mechanics.



The Path Integral Formulation of Your Life

Courtesy the Artist, Theoretical Physicist Gaurav Narain

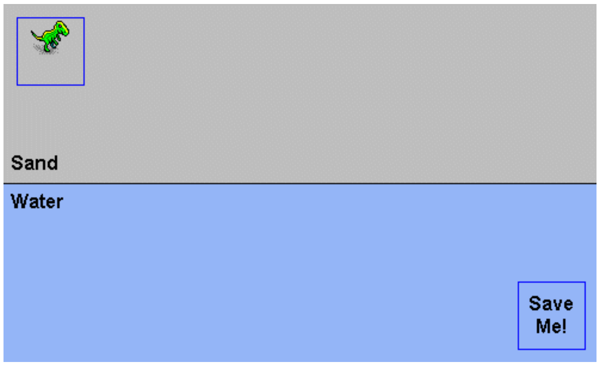
Historical Seeds for Feynman's Analysis



1. Hero (or Heron) of Alexandra (ca. year 50) - light travels from one point to another taking the shortest possible time

Left Photo: Scattering Dust in Path of Laser Beam.

2. Pierre de Fermat (1601-1665) - light travels the path taking the least time (more than one medium included)



$$t = \sum \frac{1}{v_i} s_i$$

$$t = \frac{1}{c} \sum n_i s_i$$

$$t = \frac{1}{c} \int n(s)\, ds$$

3. The Bernoulli Brothers and the Brachistocrhone Problem (1696).

Johann Bernoulli (1667-1748)	Jacob (Jacques) Bernoulli (1654-1705)

The Swiss mathematician Johann Bernoulli posed the famous ***brachistochrone*** problem in 1696. It requires one to find the curve that gives the shortest (brachistos) time (chronos) of travel for a particle sliding under the influence of gravity (no friction). No one solved it in six months.

Leibniz requested Bernoulli to extend the time so that foreign mathematicians would be sure to hear about the challenge. Bernoulli and Leibniz knew that the problem could not be solved without calculus. They were specifically after Newton since the Newton-Leibniz controversy made ambiguous the true discoverer of calculus. Word about the problem finally got to Newton on January 29, 1697. Newton's biographer Conduitt reports that Newton solved the problem in one evening after a hard day's work at the mint:

"*... in the midst of the hurry of the great recoinage, did not come home till four (in the afternoon) from the Tower very much tired, but did not sleep till he had solved it, which was by four in the morning.*"

The brachistochrone problem played an important historical role in the development of the "calculus of variations." Johann Bernoulli produced an elegant solution by framing the question in terms of optics, an effective index of refraction, and Fermat's principle. L'Hospital, Leibniz, and Johann's brother Jacob also produced solutions to the brachistochrone problem.

4. Joseph-Louis Lagrange (1736-1813). Lagrange made important contributions to the subject called the "calculus of variations" at the young age of 19. The following are general equations independent of physics.

$$\int_{t_1}^{t_2} L\, dt \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0,$$

The difference between the kinetic energy and potential energy is named the Lagrangian in honor of the mathematician.

5. Pierre Louis Moreau de Maupertuis (1698-1759). Maupertuis was the mathematician who gave us the first formulation of the ***principle of least action*** in 1746.

$$S = \int L\, dt$$

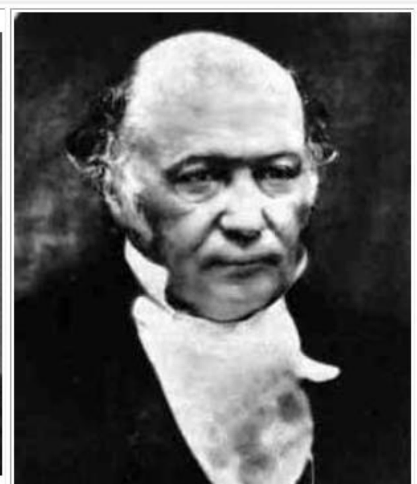
$$L = \frac{1}{2} m v^2 - V(x)$$

The L stands for Lagrangian.

6. Euler, Lagrange, and Hamilton did further work on the least-action principle of Maupertuis. Thus we hear about the Euler-Lagrange equations or Hamilton's principle of least action.

Leonhard Euler (1707-1783)

Joseph-Louis Lagrange (1736-1813)



William Rowan Hamilton (1805-1865)

Y2. Huygens-Fresnel Principle.

Christiaan Huygens (1629-1695).
Painting by Bernard Vaillant
Museum Hofwijck, Voorburg, The Netherlands

Huygens-Fresnel Principle says that you can replace a wave front with "baby waves" to get the wave of the future.

First consider a basic one-dimensional frozen wave written as an exponential. You can always take the real part.

$$\psi \sim e^{ikx}$$

In vacuum, we have

$$c = \lambda_0 f \ .$$

Augustin-Jean Fresnel (1788-1827).
From Frontispiece of his collected works (1866)
Image Courtesy Wikipedia

When the light enters a medium with a lower velocity

$$v = \lambda f \ ,$$

the wavelength shortens and you get refraction. The frequency stays the same. Therefore, you have a new $k = \dfrac{2\pi}{\lambda}$ but the same $\omega = 2\pi f$.

$$kx = \frac{2\pi}{\lambda} x = \frac{2\pi}{v/f} x = 2\pi f \frac{x}{v}$$

$$kx = 2\pi \frac{f}{c}\frac{c}{v} x = \frac{2\pi}{\lambda_0} nx = k_0 nx \rightarrow k_0 \sum n_i x_i$$

$$kx \rightarrow k_0 \int n(s)\, ds \qquad e^{i(kx-\omega t)} \rightarrow e^{ik_0 \int n(s)\, ds}$$

The phase for the spatial part is then given in the exponent $\psi \sim e^{ik_0 \int n(s)\, ds}$

Now we go back to Huygens-Fresnel Principle. The ***Huygens-Fresnel Principle*** says that you can replace a wave front with "baby waves" at each of the points on the crest to get the wave of the future.

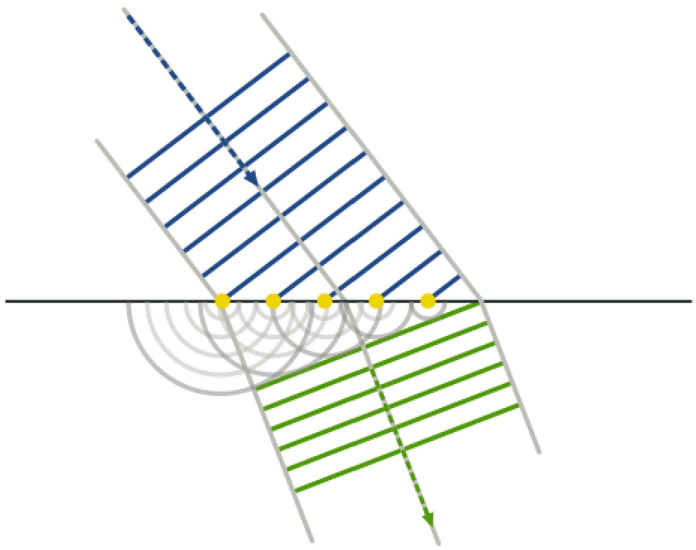


Image Courtesy Arne Nordmann, Wikipedia

First note how the wavelength shortens as the wave enters the lower medium where the speed is slower. Note the bend, i.e., the refraction. Also note that the number of crests passing per second is the same - the same frequency. Otherwise you would be creating extra wave crests.

The wavelets shown here at the boundary, are each expanding on its own, getting you the future crest.

We write this among friends in a simplified way as follows.

$$\psi(x, t+\varepsilon) = \int G(x, x')\psi(x', t)\, dx'$$

The G function is your basic point response, i.e., an emerging wavelet at each point. You add all these up and get a wave a little later and so on. The basic point response is related to our

$$G \sim e^{ik_0 \int n(s)\, ds} .$$

You evaluate G over the short time interval ε .

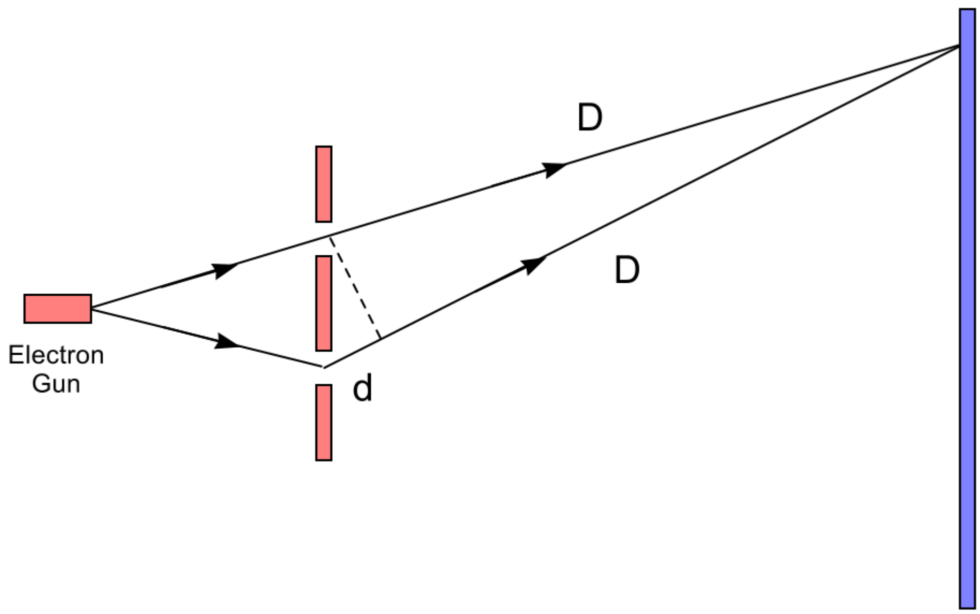
Y3. Phase in Quantum Mechanics. Phase in quantum mechanics is given by ϕ in the exponential of a wave function. An overall phase does not affect probability.

$$\psi \sim e^{i\phi}$$

Consider the electron beam going through the double slit. The relative phase is given by

$$\Delta\phi = kd = \frac{\hbar kd}{\hbar} = \frac{pd}{\hbar}$$ using de Broglie's relation $p = \hbar k$, i.e., $p = \frac{h}{\lambda}$.

Now we consider the action $S = \int L\,dt$ for each path and subtract them.



The time for the path from the slits to the screen takes place during the same time interval t.

$$S_1 = L_1 t = \frac{1}{2} m(\frac{D}{t})^2 t = \frac{mD^2}{2t} \qquad S_2 = L_2 t = \frac{m(D+d)^2}{2t} \approx \frac{m(D^2 + 2Dd)}{2t}$$

$$\Delta S = S_2 - S_1 = \frac{mDd}{t} = mvd = pd$$

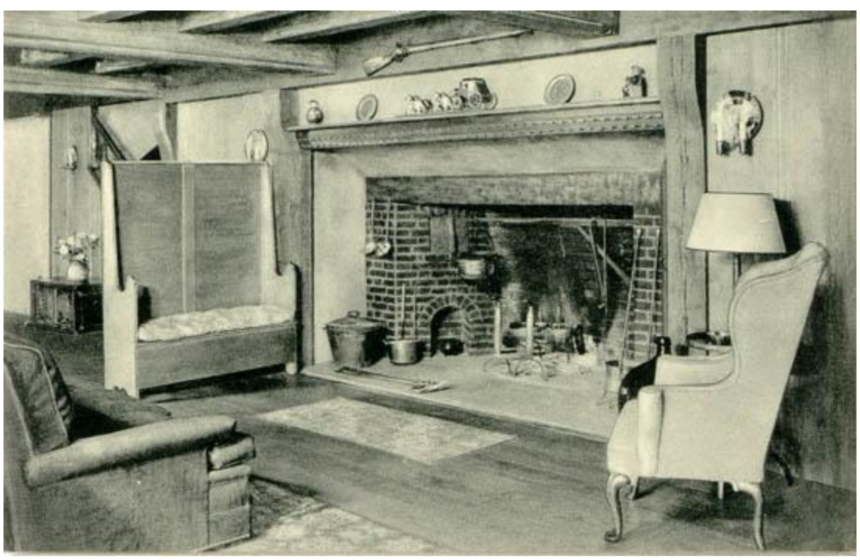
Remember $\Delta\phi = \frac{pd}{\hbar}$ from above? Therefore, $\Delta\phi = \frac{\Delta S}{\hbar}$ and $\phi = \frac{S}{\hbar}$.

Y4. Dirac's Analogy. Dirac published an observation in 1933 that there is an analogy between the phase in optics and the phase in quantum mechanics.

Paul Adrien Maurice Dirac
1902-1984

Optics	Quantum Mechanics Analogy
$G \sim e^{ik_o S}$	$e^{iS/\hbar}$
$S = \int n(s)\, ds$	$S = \int L(t)\, dt$
Minimize Time	Minimize Action
Fermat's Principle	Hamilton's Principle

Note that both **S** and **h** have units of action, i.e., energy * time. Therefore, $S/\hbar$ is dimensionless. In the case of optics, the wave number k_o (in vacuum) does the trick to obtain a dimensionless quantity for the exponent.



Feynman was at the Nassau tavern, (left, postcard image), a year before he received his Ph.D. from Princeton, under thesis advisor Wheeler (Bohr's student). Feynman tells the story.

"... but when I was struggling with this problem, I went to a beer party in the Nassau Tavern in Princeton. There was a gentleman, newly arrived from Europe (Herbert Jehle) who came and sat next to me. Europeans are much more serious than we are in America because they think that a good place to discuss intellectual matters is a beer party. So, he sat by me and asked, 'What are you doing' and so on, and I said, 'I'm drinking beer.' Then I realized that he wanted to know what work I was doing and I told him I was struggling with this problem, and I simply turned to him and said, 'Listen, do you know any way of doing quantum mechanics, starting with action - where the action integral comes into the quantum mechanics?' 'No,' he said, 'but Dirac has a paper in which the Lagrangian, at least, comes into quantum mechanics. I will show it to you tomorrow.'

"Next day we went to the Princeton Library, they have little rooms on the side to discuss things, and he showed me this paper. What Dirac said was the following: There is in quantum mechanics a very important quantity which carries the wave function from one time to another, ... " ***Feynman, Nobel Lecture, December 11, 1965.***

Herbert Jehle (1907-1983)
Courtesy The George Washington University
Washington, DC

"Professor Jehle showed me this, I read it, he explained it to me, and I said, 'What does he mean, they are analogous; what does that mean, analogous? What is the use of that?'

"He said, 'You Americans! You always want to find a use for everything!'

"I said, that I thought that Dirac must mean that they were equal. 'No,' he explained, 'he doesn't mean they are equal.' 'Well,' I said, 'Let's see what happens if we make them equal.'"

So Feynman takes Dirac's analogy as more than an analogy and writes

$$\psi(x,t+\varepsilon)=\int G(x,x')\psi(x',t)\,dx'$$

with

$$G=e^{iS/\hbar} \quad \text{and} \quad S=\int_t^{t+\varepsilon} L\,dt \text{ over the time interval } \varepsilon .$$

Taking Dirac literally, Feynman sets $G=e^{iS/\hbar}$ and writes the above Green's function type equation. In our next section we will see what Feynman found.

Y5. Feynman's Derivation of the Schrödinger Equation.

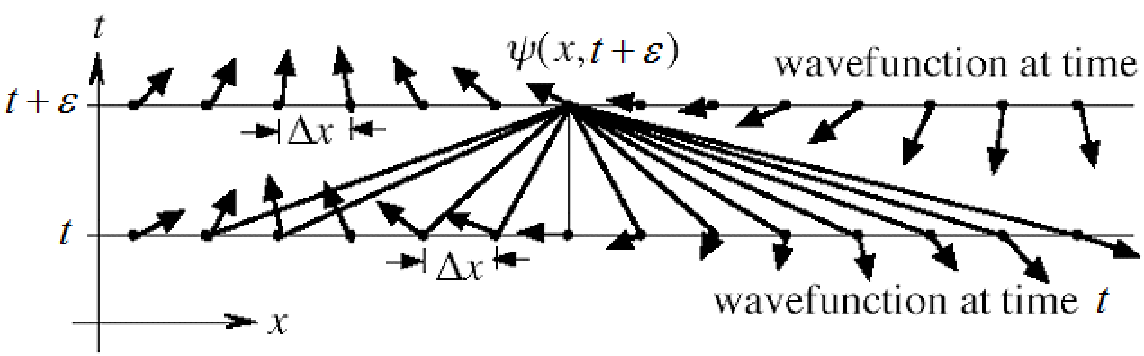
Richard Feynman (1918-1988)
Photo Courtesy nobelprize.org

"'Well,' I said, 'Let's see what happens if we make them equal.'" ***Feynman, Nobel Lecture, 1965.*** Thus Feynman wanted to see what would happen if he took the following to be true.

$$\psi(x,t+\varepsilon) = \int G(x,x')\psi(x',t)\,dx'$$

$$G = e^{iS/\hbar} \qquad S = \int_t^{t+\varepsilon} L\,dt$$

From the Huygens' principle of optics, one arrives at the new wave front at time $t+\varepsilon$ from wavelets emanating at an earlier time t. Dirac's analogy with quantum mechanics suggests a wild idea. Since the wave function represents a probability distribution, the integral can be interpreted as summing over all the possible paths from time t to time $t+\varepsilon$ to arrive at x. The direct paths are shown below with the complex value of the wave function at each point represented by an arrow. Remember that a complex number can be represented as a modulus and direction, i.e., a ***phasor***.



Courtesy Thomas A. Moore

One sums over the different paths starting from any **x'** and arriving at **x** within the same time frame. The contributions from the various paths are weighted by the exponential factor which carries the action for each path. In the classical limit, all the many crazy paths have to cancel, i.e., interfere destructively, leaving the one classical path.

Feynman considered a small **dt** since he knew that the principle of least action is true no matter what the time interval is. This is the easier case to consider.

References: David Derbes, "Feynman's Derivation of the Schrödinger Equation," *American Journal of Physics* **64**, 881-884 (1996) and Masturi Ulul Amri, "Formalizing Feynman's Derivaion of the Schrödinger Equation," ICMSE 2015 International Conference on Mathematics, Science, and Education.

$$\psi(x,t+\varepsilon)=\int G(x,x')\psi(x',t)\,dx' \qquad G=Ae^{iS/\hbar}$$

Note that we include the proportionality constant A. When Feynman did his calculation, he tried with A = 1 since from Dirac's comment he though the Green's function might be equal to the phase term. He found that was not the case, as we will show.

$$S=\int_t^{t+\varepsilon} L\,dt \qquad S=\int_t^{t+\varepsilon} L\,dt \approx \overline{L}\,\varepsilon \qquad \overline{L}=\overline{K}-\overline{V}$$

$$\overline{K}=\frac{1}{2}m\left(\frac{\Delta x}{\Delta t}\right)^2=\frac{1}{2}m\frac{(x-x')^2}{\varepsilon^2}$$ as we go from x' to x during time ε .

$$\overline{V}=V\left(\frac{x+x'}{2}\right) \qquad S\approx\overline{L}\,\varepsilon=\frac{1}{2}m\frac{(x-x')^2}{\varepsilon^2}\varepsilon-V\left(\frac{x+x'}{2}\right)\varepsilon$$

$$G(x,x')=Ae^{iS/\hbar}\approx A\exp\left[\frac{im(x-x')^2}{2\hbar\varepsilon}-V\left(\frac{x+x'}{2}\right)\frac{i\varepsilon}{\hbar}\right]$$

$$G(x,x')\approx A\exp\left[\frac{im(x-x')^2}{2\hbar\varepsilon}\right]\exp\left[-V\left(\frac{x+x'}{2}\right)\frac{i\varepsilon}{\hbar}\right]$$

Since ε is very small, we can do a Taylor series expansion for the second factor.

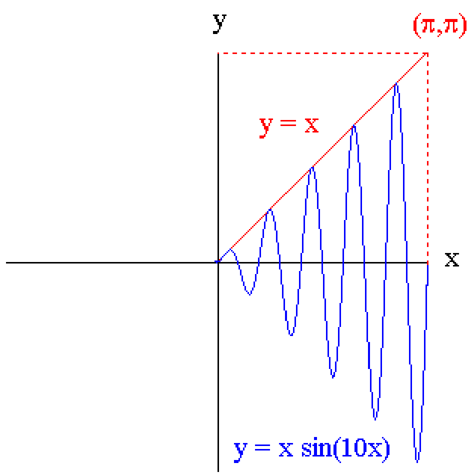
$$G(x,x') \approx A\exp\left[\frac{im(x-x')^2}{2\hbar\varepsilon}\right]\left[1-V(\frac{x+x'}{2})\frac{i\varepsilon}{\hbar}\right]$$

$$\psi(x,t+\varepsilon) = \int G(x,x')\psi(x',t)\,dx'$$

$$\psi(x,t+\varepsilon) \approx A\int_{-\infty}^{\infty} e^{\frac{im(x-x')^2}{2\hbar\varepsilon}}\left[1-V(\frac{x+x'}{2})\frac{i\varepsilon}{\hbar}\right]\psi(x',t)dx'$$

Now in the sum of all paths, it seems reasonable that the most contribution will come from x close to x’. As the system evolves, doesn’t it make sense that a path from a nearby neighbor will contribute more than a path very far like from New York to London? Feynman gives a clever argument with oscillations that we will not delve into here. We will see shortly that everything will work out nicely.

Feynman reasoned as follows, but I will give you an argument shortly involving a Dirac delta function that is conceptually much easier to grasp. So you can skip the oscillation discussion here if you want. Since $\hbar\varepsilon$ is extremely small, then $\frac{1}{\hbar\varepsilon}$ is very large. This mean rapid oscillations due the exponent since Euler's relation indicates sines and cosines. This will give very thin positive and negative slices that will tend to cancel out.



Using Zona Land's Graphics Calculator

Check out the integration of $x\sin(10x)$. See how the rapid oscillations give plus and minus areas that are roughly equal in magnitude.

Feynman reasons that only for very small x-x' do things matter. That's where your get something for your integration.

So Feynman defines a new variable. We will call it η: $\eta = x - x'$. Note $d\eta = -dx'$.

The limits of integration get flipped and a minus sign is brought in with the differential. We flip back the integration and take the minus sign away. Then, our original

$$\psi(x,t+\varepsilon) \approx A\int_{-\infty}^{\infty} e^{\frac{im(x-x')^2}{2\hbar\varepsilon}} \left[1 - V(\frac{x+x'}{2})\frac{i\varepsilon}{\hbar}\right]\psi(x',t)dx'$$

with $x - x' = \eta$, $x' = x - \eta$, and $x + x' = x + (x-\eta) = 2x - \eta$, brings us to

$$\psi(x,t+\varepsilon) \approx \int_{\infty}^{-\infty} e^{\frac{im\eta^2}{2\hbar\varepsilon}} \left[1 - V(x-\frac{\eta}{2})\frac{i\varepsilon}{\hbar}\right]\psi(x-\eta,t)(-d\eta) .$$

We can flip the integration limits and lose the minus sign in front of the differential.

$$\psi(x,t+\varepsilon) \approx A\int_{-\infty}^{\infty} e^{\frac{im\eta^2}{2\hbar\varepsilon}} \left[1 - V(x-\frac{\eta}{2})\frac{i\varepsilon}{\hbar}\right]\psi(x-\eta,t)d\eta$$

As we continue, you can see why Feynman is called a "magician" when it comes to theoretical physics. Continuing, since things matter only when η is small, we expand the wave function in the integrand.

$$\psi(x-\eta,t) = \psi(x,t) - \frac{\partial\psi(x,t)}{\partial x}\eta + \frac{1}{2}\frac{\partial^2\psi(x,t)}{\partial x^2}\eta^2 - \ldots$$

Similarly, ε is small, so we expand the wave function on the left side in terms of time..

$$\psi(x,t+\varepsilon) = \psi(x,t) - \frac{\partial\psi(x,t)}{\partial t}\varepsilon + \frac{1}{2}\frac{\partial^2\psi(x,t)}{\partial t^2}\varepsilon^2 - \ldots$$

Now Feynman realized that his first guess where A = 1 cannot be true. He recognized that

$$\psi(x,t+\varepsilon) \neq \int_{-\infty}^{\infty} e^{\frac{im\eta^2}{2\hbar\varepsilon}} \left[1 - V(x-\frac{\eta}{2})\frac{i\varepsilon}{\hbar}\right]\psi(x-\eta,t)d\eta$$

since to zeroth order

$$\psi(x,t) \neq \int_{-\infty}^{\infty} e^{\frac{im\eta^2}{2\hbar\varepsilon}} \psi(x,t) d\eta = \psi(x,t) \int_{-\infty}^{\infty} e^{\frac{im\eta^2}{2\hbar\varepsilon}} d\eta \; .$$

Feynman evaluates the integral using $\int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}}$ and doesn't care that there is an imaginary number in the exponent and that the exponent even has no minus sign. He gets

$$\psi(x,t) \int_{-\infty}^{\infty} e^{\frac{im\eta^2}{2\hbar\varepsilon}} d\eta = \sqrt{\frac{\pi 2\hbar\varepsilon}{-im}} = \sqrt{\frac{2\pi i\hbar\varepsilon}{m}}$$

So Feynman concludes $G \neq e^{iS/\hbar}$. So the next best guess is **proportional**.

$$G = Ae^{iS/\hbar} \quad \text{with} \quad A = \sqrt{\frac{m}{2\pi i\hbar\varepsilon}}$$

Bringing the proportionality constant in,

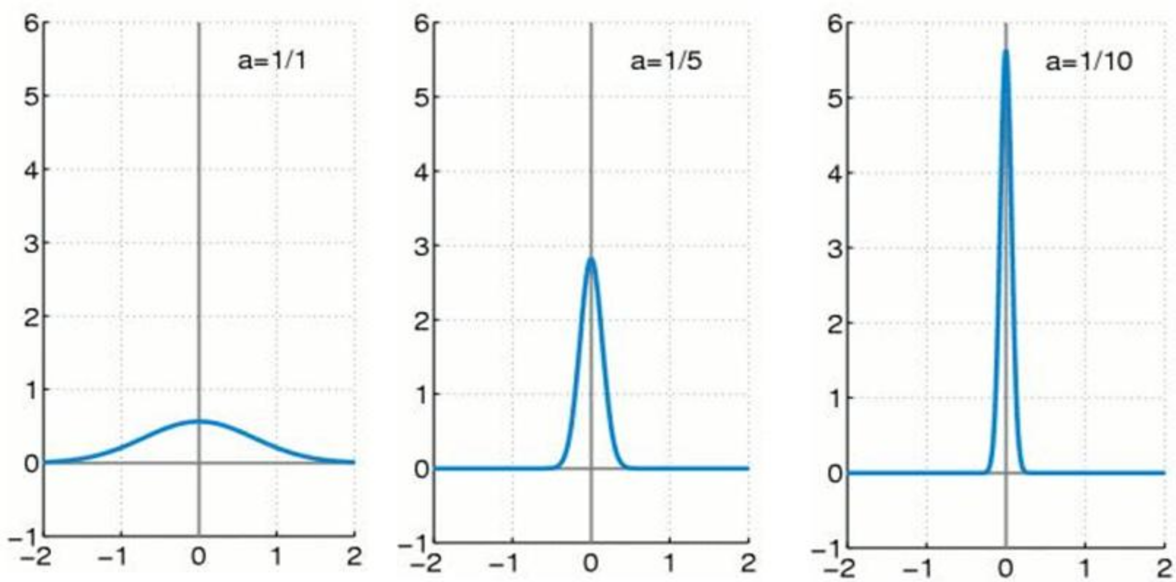
$$\psi(x,t+\varepsilon) \approx \sqrt{\frac{m}{2\pi i\hbar\varepsilon}} \int_{-\infty}^{\infty} e^{\frac{im\eta^2}{2\hbar\varepsilon}} \left[1 - V(x - \frac{\eta}{2}) \frac{i\varepsilon}{\hbar} \right] \psi(x-\eta,t) d\eta$$

But now observe something very strange. Focus your attention on $\sqrt{\frac{m}{2\pi i\hbar\varepsilon}} e^{\frac{im\eta^2}{2\hbar\varepsilon}}$.

Recall from our Dirac delta function class that we looked at the common Gaussian in statistics:

$$P(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}}$$ and the related sequence $$\delta_a(x) = \frac{1}{a\sqrt{\pi}} e^{-\frac{x^2}{a^2}}$$.

where $a^2 = 2\sigma^2$.



Courtesy Wikipedia

$$\lim_{a\to 0} \delta_a(x) = \lim_{a\to 0} \frac{1}{a\sqrt{\pi}} e^{-\frac{x^2}{a^2}} = \delta(x)$$

Now consider the following as sequence with epsilon getting smaller.

$$\delta_\varepsilon = \sqrt{\frac{m}{2\pi i\hbar\varepsilon}}\, e^{\frac{im\eta^2}{2\hbar\varepsilon}}$$

Make the assignment $\dfrac{1}{a} = \sqrt{\dfrac{m}{2i\hbar\varepsilon}}$.

Then the factor in front of the exponential becomes

$$\sqrt{\frac{m}{2\pi i\hbar\varepsilon}} = \frac{1}{a\sqrt{\pi}} .$$

Note $\frac{1}{a^2} = \frac{m}{2i\hbar\varepsilon} = \frac{im}{2i^2\hbar\varepsilon} = -\frac{im}{2\hbar\varepsilon}$ and $-\frac{1}{a^2} = \frac{m}{2i\hbar\varepsilon} = \frac{im}{2i^2\hbar\varepsilon} = \frac{im}{2\hbar\varepsilon}$.

What? We have a delta sequence if we do not get intimated by the appearance of the imaginary number below and the fact that we see no explicit minus sign in the exponent!

$$\delta_\varepsilon = \sqrt{\frac{m}{2\pi i\hbar\varepsilon}}\, e^{\frac{im\eta^2}{2\hbar\varepsilon}} = \delta_\varepsilon(\eta)$$

So epsilon approaches zero as it will, then the following

$$\psi(x,t+\varepsilon) \approx \sqrt{\frac{m}{2\pi i\hbar\varepsilon}} \int_{-\infty}^{\infty} e^{\frac{im\eta^2}{2\hbar\varepsilon}} \left[1 - V(x-\frac{\eta}{2})\frac{i\varepsilon}{\hbar}\right] \psi(x-\eta,t)\,d\eta$$

approaches

$$\psi(x,t+\varepsilon) \approx \int_{-\infty}^{\infty} \delta(\eta) \left[1 - V(x-\frac{\eta}{2})\frac{i\varepsilon}{\hbar}\right] \psi(x-\eta,t)\,d\eta$$

and we now have the proof that contributions will occur when ε is nonzero and small for only η close to zero. In other words, contributions come from x' close to x, the nearby neighbors. Our discussion with the Dirac delta function is another form of Feynman's oscillation argument.. The Dirac delta function argument is easier for us since we have examined delta sequences before.

The potential can be further simplified by expanding to first order in η ,

$$V(x-\frac{\eta}{2}) \approx V(x) - V'(x)\frac{\eta}{2} .$$

But when the ε is included,

$$V(x-\frac{\eta}{2})\varepsilon \approx V(x)\varepsilon - V'(x)\frac{\eta}{2}\varepsilon \approx V(x)\varepsilon$$,

since the $\eta\varepsilon$ product of two small quantities can be neglected.

$$\psi(x,t+\varepsilon) \approx A\int_{-\infty}^{\infty} e^{\frac{im\eta^2}{2\hbar\varepsilon}} \left[1 - V(x)\frac{i\varepsilon}{\hbar}\right] \psi(x-\eta,t)\,d\eta$$

Now we go back to insert

$$\psi(x-\eta,t) = \psi(x,t) - \frac{\partial\psi(x,t)}{\partial x}\eta + \frac{1}{2}\frac{\partial^2\psi(x,t)}{\partial x^2}\eta^2 - \ldots$$

There are three integrals to do.

Summary:

$$\psi(x,t+\varepsilon) \approx A\int_{-\infty}^{\infty} e^{\frac{im\eta^2}{2\hbar\varepsilon}} \left[1 - V(x)\frac{i\varepsilon}{\hbar}\right] \psi(x-\eta,t)\,d\eta$$

$$\psi(x-\eta,t) = \psi(x,t) - \frac{\partial\psi(x,t)}{\partial x}\eta + \frac{1}{2}\frac{\partial^2\psi(x,t)}{\partial x^2}\eta^2 - \ldots$$

$$I_1 = \psi(x,t)A\int_{-\infty}^{\infty} e^{\frac{im\eta^2}{2\hbar\varepsilon}} \left[1 - V(x)\frac{i\varepsilon}{\hbar}\right] d\eta$$. Note: to 1st order in ε .

$$I_2 = -\frac{\partial \psi(x,t)}{\partial x} A \int_{-\infty}^{\infty} e^{\frac{im\eta^2}{2\hbar\varepsilon}} \eta d\eta$$, neglecting the $\varepsilon\eta$ term.

$$I_3 = \frac{1}{2} \frac{\partial^2 \psi(x,t)}{\partial x^2} A \int_{-\infty}^{\infty} e^{\frac{im\eta^2}{2\hbar\varepsilon}} \eta^2 d\eta$$, neglecting the $\varepsilon\eta^2$ term.

For the first integral,

$$I_1 = \left[1 - V(x)\frac{i\varepsilon}{\hbar}\right]\psi(x,t) A \int_{-\infty}^{\infty} e^{\frac{im\eta^2}{2\hbar\varepsilon}} d\eta = \left[1 - V(x)\frac{i\varepsilon}{\hbar}\right]\psi(x,t)$$

since $\int_{-\infty}^{\infty} e^{\frac{im\eta^2}{2\hbar\varepsilon}} d\eta = \frac{1}{A}$. $\quad I_1 = \left[1 - V(x)\frac{i\varepsilon}{\hbar}\right]\psi(x,t)$

The second integral is zero since we have an even function multiplied by an odd one.

$$I_2 \approx -\frac{\partial \psi(x,t)}{\partial x} A \int_{-\infty}^{\infty} e^{\frac{im\eta^2}{2\hbar\varepsilon}} \eta d\eta = 0$$

But you might say we have oscillations: sines and cosines? Aren't sine functions odd. Let's see.

$$e^{iu} = \cos u + i \sin u$$ (an even function and an odd function).

But we do not have this case. We have the next case.

$$e^{iu^2} = \cos u^2 + i \sin u^2$$ (Now both functions on the right are even!)

The third integral is

$$I_3 = \frac{1}{2}\frac{\partial^2\psi(x,t)}{\partial x^2} A\int_{-\infty}^{\infty} e^{\frac{im\eta^2}{2\hbar\varepsilon}}\eta^2 d\eta \quad .$$

Feynman was fond of doing integrals with the derivative trick. We have seen the derivative trick throughout this course.

$$\int_{-\infty}^{\infty}\eta^2 e^{-a\eta^2}dx = -\frac{d}{d\alpha}\int_{-\infty}^{\infty}e^{-a\eta^2}dx = -\frac{d}{d\alpha}\sqrt{\frac{\pi}{\alpha}} = \frac{1}{2\alpha}\sqrt{\frac{\pi}{\alpha}}$$

$$I_3 = \frac{1}{2}\frac{\partial^2\psi(x,t)}{\partial x^2} A\frac{1}{2}\left[\frac{2\hbar\varepsilon}{-im}\right]\frac{1}{A}$$

$$I_3 = \frac{1}{2}\frac{\partial^2\psi(x,t)}{\partial x^2}\frac{1}{2}\left[\frac{2i\hbar\varepsilon}{m}\right]$$

$$I_3 = \frac{1}{2}\frac{\partial^2\psi(x,t)}{\partial x^2}\left[\frac{i\hbar\varepsilon}{m}\right]$$

$$I_3 = \frac{\hbar}{2m}\frac{\partial^2\psi(x,t)}{\partial x^2}i\varepsilon$$

$$\psi(x,t+\varepsilon) \approx I_1 + I_2 + I_3$$

$$I_1 = \left[1 - V(x)\frac{i\varepsilon}{\hbar}\right]\psi(x,t) \qquad I_2 = 0 \qquad I_3 = \frac{\hbar}{2m}\frac{\partial^2\psi(x,t)}{\partial x^2}i\varepsilon$$

$$\psi(x,t+\varepsilon) \approx \left[1 - V(x)\frac{i\varepsilon}{\hbar}\right]\psi(x,t) + \frac{\hbar}{2m}\frac{\partial^2\psi(x,t)}{\partial x^2} i\varepsilon$$

$$\psi(x,t+\varepsilon) \approx \psi(x,t) - \frac{i\varepsilon}{\hbar}V(x)\psi(x,t) + \frac{\hbar}{2m}\frac{\partial^2\psi(x,t)}{\partial x^2} i\varepsilon$$

$$\psi(x,t+\varepsilon) - \psi(x,t) \approx -\frac{i\varepsilon}{\hbar}V(x)\psi(x,t) + \frac{\hbar}{2m}\frac{\partial^2\psi(x,t)}{\partial x^2} i\varepsilon$$

$$\frac{\psi(x,t+\varepsilon) - \psi(x,t)}{i\varepsilon} \approx -\frac{1}{\hbar}V(x)\psi(x,t) + \frac{\hbar}{2m}\frac{\partial^2\psi(x,t)}{\partial x^2}$$

$$-i\frac{\psi(x,t+\varepsilon) - \psi(x,t)}{\varepsilon} \approx \frac{\hbar}{2m}\frac{\partial^2\psi(x,t)}{\partial x^2} - \frac{1}{\hbar}V(x)\psi(x,t)$$

$$-i\frac{\partial\psi(x,t)}{\partial t} \approx \frac{\hbar}{2m}\frac{\partial^2\psi(x,t)}{\partial x^2} - \frac{1}{\hbar}V(x)\psi(x,t)$$

$$\boxed{i\hbar\frac{\partial\psi(x,t)}{\partial t} \approx -\frac{\hbar^2}{2m}\frac{\partial^2\psi(x,t)}{\partial x^2} + V(x)\psi(x,t)}$$

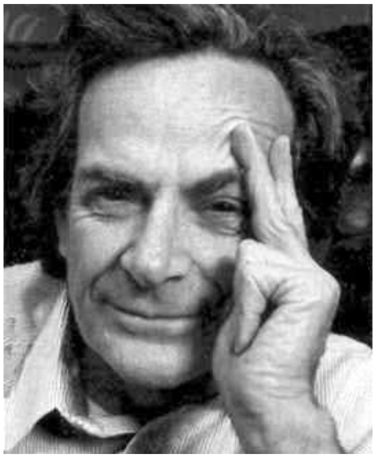
The Schrödinger equation gives the evolution of the state!

See next page for interesting photos.

Feynman Photo Courtesy Caltech
Jehle Photo Courtesy The George Washington University
Dirac-Feynman Photo Courtesy Caltech

"So, I turned to Professor Jehle, not really understanding, and said, 'Well, you see Professor Dirac meant that they were proportional.'

Richard Feynman (1918-1988)



Professor Jehle's eyes were bugging out - he had taken out a little notebook and was rapidly copying it down from the blackboard, and said, 'No, no, this is an important discovery. You Americans are always trying to find out how something can be used. That's a good way to discover things!'

Herbert Jehle (1907-1983)

So, I thought I was finding out what Dirac meant, but, as a matter of fact, had made the discovery that what Dirac thought was analogous, was, in fact, equal. I had then, at least, the connection between the Lagrangian and quantum mechanics, but still with wave functions and infinitesimal times." ***Feynman, Nobel Lecture, 1965***

Dirac and Feynman at a Relativity Conference in Warsaw, Poland (July 1962).

Feynman and Dirac were at a bicentennial celebration in the Fall of 1946. Feynman wanted to know what Dirac had meant by "analogous" in that historical 1933 paper.

"Did you know that they were proportional?" asked Feynman.

"Are they?" Dirac inquired.

"Yes," said Feynman.

"Oh, that's interesting," was Dirac's final comment.