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Hearing the transformation of conical to closed-pipe resonances

Michael J Ruiz

Department of Physics, University of North Carolina at Asheville, Asheville, NC, United States of America

E-mail: mjtruiz@gmail.com

Abstract

The harmonics for an open cone with slant length L are the same as the harmonics for an open pipe with length L. When the cone is transformed through phases of closed–open conical frusta into a cylinder of length L closed at one end, the fundamental halves and only odd harmonics remain. A simple approach using boundary conditions is presented in order to understand this remarkable fact. A new free interactive HMTL5 application is provided which enables the user to hear the resonances of a complete cone transform into the pitches of a conical frustum closed at the small end, and eventually into the odd harmonics of a closed cylindrical pipe.

Background

In the standard approach to standing waves (stationary waves) in open pipes, n half-waves are fitted to the pipe length L, where n is 1, 2, 3, … etc. For a pipe closed at one end, i.e. a closed pipe, m quarter-waves are fitted to L, where m is an odd number 1, 3, 5, … etc. These procedures are described by the following formulas.

$$\text{Open pipe} \quad n\frac{\lambda}{2} = L \quad n = 1, 2, 3 \ldots \tag{1a}$$

$$\text{Closed pipe} \quad m\frac{\lambda}{4} = L \quad m = 1, 3, 5 \ldots \tag{1b}$$

The first standing wave, the fundamental, is shown for an open pipe and closed pipe in figure 1. The transverse sinusoidal curves indicate the changes in pressure relative to equilibrium pressure at each position along the length of the pipe for the longitudinal standing wave. The relative pressure referenced to atmospheric pressure is called the gauge pressure. At an open end, the pressure is equal to atmospheric pressure and therefore the gauge pressure is zero. Such a location is called a pressure node. In contrast, a pressure antinode is found where maximum pressure changes occur, e.g. at a closed end.

An interesting consideration often neglected in texts is a reasonable restriction on the length L compared to the diameter d of the pipe. Surely, if L gets too small, the pipe becomes a circular band and the standing waves are lost. Students can be encouraged to make an estimate here. Someone might suggest $d \ll L$ or $d \ll \lambda$. The condition $d \ll \lambda$ for the closed pipe leads to $d \ll \frac{4L}{m}$. Since this paper investigates several harmonics, we use $d \ll \frac{4L}{10}$ as an order of magnitude estimate. Taking one-half this limit gives a reasonable bound where the 'much less than' sign is replaced merely by a 'less than' sign:

$$d < \frac{L}{5}. \tag{2}$$

Such an exercise in reasoning develops critical thinking skills.

Finally, equations (1*a*) and (1*b*) can be expressed in terms of frequencies using the wave relation $v = \lambda f$, where v is the speed of sound and f is the frequency of the wave. The resulting frequencies are called harmonics.

$$\text{Open pipe} \quad f_n = \frac{nv}{2L} \quad n = 1, 2, 3 \ldots \tag{3a}$$

$$\text{Closed pipe} \quad f_m = \frac{mv}{4L} \quad m = 1, 3, 5 \ldots \tag{3b}$$

The approach using boundary conditions

Imposing boundary conditions at each pipe end leads to an analysis of conical pipes accessible to introductory students. It is instructive to first reproduce the results in the last section with the new approach before proceeding to the more complicated conical frustum. The simplest periodic functions are sines and cosines: $A\ \sin(kx)$ and $C\ \cos(kx)$, where the wave number $k = \frac{2\pi}{\lambda}$. We use the constant C for the cosine since we are reserving B for another purpose in this paper. The sine wave is 'nature's wave,' pervasive everywhere. The sines and cosines describe oscillations on springs, the swinging pendulum, waves on strings, electromagnetic waves, atomic vibrations and much more. Here these periodic functions are applied to pipes.

For the open pipe, take $p(x) = A\ \sin(kx)$ as the profile of the pressure wave so that $p(0) = 0$, i.e. the gauge pressure must be zero at the left open end. Note that a suitable oscillatory time factor such as $\cos(\omega\, t)$ is left out so the focus is on the pressure profile function. This boundary condition at the left open end, where $x = 0$, eliminates the inclusion of $C\ \cos(kx)$. The boundary condition at the right open end requires $p(L) = A\ \sin(kL) = 0$. This second boundary condition leads to $kL = n\pi$, where $n = 1, 2, 3 \ldots$ Substituting $k = \frac{2\pi}{\lambda}$, gives $\frac{2\pi}{\lambda}L = n\pi$, which is equivalent to equation (1a), $n\frac{\lambda}{2} = L$.

For the closed pipe, take the pressure profile function to be $p(x) = C\ \cos(kx)$ so that maximum pressure variation occurs at the closed left end, i.e. $p(0) = C$. The boundary condition at the right open end is $p(L) = C\ \cos(kL) = 0$. This second boundary condition leads to $kL = \frac{m}{2}\pi$, where $m = 1, 3, 5 \ldots$, giving equation (1b), $m\frac{\lambda}{4} = L$.

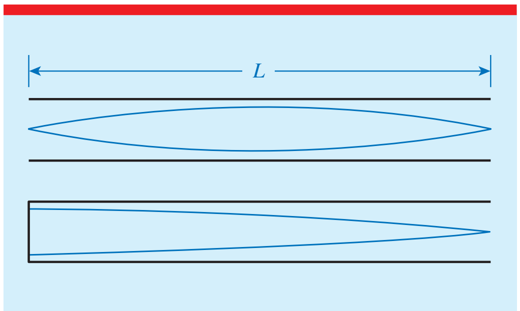

Figure 1. Longitudinal fundamentals for an open pipe (top) and a closed pipe (bottom). The curves indicate the pressure changes relative to equilibrium pressure for the longitudinal standing waves.

The conical frustum and transcendental equations

The study of resonances in conical structures dates back to Nobel Laureate John William Strutt (1842–1919), known as Lord Rayleigh. Rayleigh works out the solution using the 3D wave equation with spherical coordinates in the second edition (1896) of his masterpiece *The Theory of Sound* [1]. Rayleigh's solution can be written down without recourse to differential equations by modifying the sine and cosine functions encountered earlier with the replacement of kr for kx and dividing by r. The general solution is

$$p(r) = \frac{A\ \sin(kr)}{r} + \frac{C\ \sin(kr)}{r}, \tag{4}$$

where the coordinate r is measured from the apex of the conical configuration outward along the slanted edge of the cone. Refer to figure 2.

The radial coordinate is dictated by the symmetry of the cone with the coordinate r measured from the origin at the tip of the cone. The $\frac{1}{r}$ factor is needed due to the inverse-square law for the energy as the wave travels outward in three dimensions. Since the energy is proportional to the square of the amplitude, $\frac{1}{r}$ must appear in the amplitude equation. The sketch in figure 2 illustrates a sliced conical pipe closed at the smaller left end and open at the larger right end. The choice of closed small end with an open large end is made to model conical bores in musical instruments. Two examples are the saxophone and tuba,

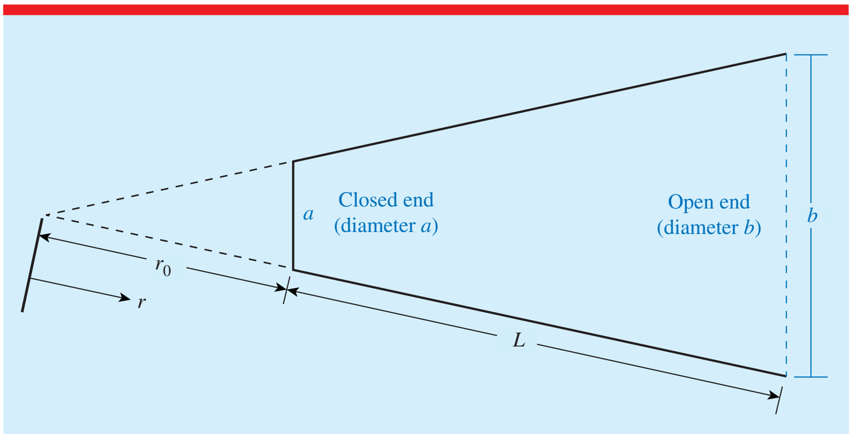

Figure 2. Conical frustum closed at the smaller left end where the pipe diameter is a and open at the larger right end where the pipe diameter is b. The slanted length of the conical pipe is L. The figure is not drawn to scale since b should be less than $\frac{L}{5}$ as explained in the first section of this paper.

where the mouthpiece serves as the small closed end and the bell flare is attached to the open end.[1]

Boundary conditions need to be set for figure 2 at the closed end ($r = r_0$) and the open end ($r = r_0 + L$). For the right boundary condition at the open end, the gauge pressure

$$p(r_0 + L) = 0. \tag{5}$$

The following approach is along the lines of the work of Ayers, Eliason, and Mahgerefteh [2]. Choosing

$$p(r) = \frac{A \sin[k(r_0 + L - r)]}{r} \tag{6}$$

satisfies this boundary condition for a gauge pressure node at $r = r_0 + L$, i.e. the open right end. For the closed end at the left,

$$\left.\frac{\mathrm{d}p(r)}{\mathrm{d}r}\right|_{r = r_0} = 0 \tag{7}$$

since for the pressure antinode, the profile of the gauge pressure function must be a maximum (extremum), as seen for the closed pipe in figure 1. The details of working out equation (7) are found in the appendix. The result is the transcendental equation

$$\tan\left[\pi\left(\frac{f}{f_0}\right)\right] = -\left[\frac{B}{1 - B}\right]\pi\left(\frac{f}{f_0}\right), \tag{8}$$

[1] The flare is joined to the large open end of the conical bore.

where $f_o = \frac{v}{2L}$, the fundamental for an open pipe of length L and $B \equiv \frac{a}{b}$, the ratio of the smaller diameter at the closed end to the larger diameter at the open end (see figure 2). Equation (8) is a transcendental equation which will shortly be solved numerically. But first consider the two interesting limiting cases: the complete cone ($a = 0$) and the closed cylindrical pipe ($a = b$).

The limiting cases: the complete cone and the closed cylindrical pipe

Now come two big surprises. In the limiting case of a complete cone ($a = 0$), the parameter $B \equiv \frac{a}{b} = 0$ and equation (8) reduces to

$$\tan\left[\pi\left(\frac{f}{f_0}\right)\right] = 0. \tag{9}$$

The solutions are $\pi\left(\frac{f}{f_0}\right) = n\pi$, which immediately leads to the astonishing result

$$f_n = nf_0 = \frac{nv}{2L}, \tag{10}$$

equation (3a) for the open pipe. Therefore, the conical pipe of slant length L has the identical harmonic series as an open cylindrical pipe of length L.

Now consider the limiting case where the closed conical frustum turns into a cylindrical closed pipe ($a = b$). Since this condition leads to $B \equiv \frac{a}{b} = 1$, we are careful to take a limit to avoid dividing by zero:

$$\tan\left[\pi\left(\frac{f}{f_0}\right)\right] = -\lim_{B\to 1}\left[\frac{B}{1-B}\right]\pi\left(\frac{f}{f_0}\right). \tag{11}$$

The tangent function approaches negative infinity at $\frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}$ and so on for positive values of f. Therefore, $\pi\left(\frac{f}{f_0}\right) = m\frac{\pi}{2}$ where $m = 1, 3, 5 \ldots$ This equation leads to the odd series of harmonics for a closed cylindrical pipe of length L with fundamental $\frac{f_0}{2}$,

$$f_m = m\frac{f_0}{2} = \frac{mv}{4L} \quad \text{with} \quad m = 1, 3, 5 \ldots, \tag{12}$$

which is equation (3b) encountered earlier for the closed pipe. This result is the second big surprise.

How can the harmonic series of all harmonics (complete cone) transform into a series of only odd harmonics (cylindrical closed pipe)? Let the harmonic sequence H_{cone} be 200 Hz, 400 Hz, 600 Hz, 800 Hz, 1000 Hz, etc. Equation (12) indicates that for the cylindrical closed pipe, the starting fundamental frequency is one half that for the open pipe. In musical terms, the fundamental for the cylindrical closed pipe is an octave lower. Consider subtracting 100 Hz from each frequency in the sequence H_{cone}. The new series is 100 Hz, 300 Hz, 500 Hz, 700 Hz, 900 Hz, etc, which can be called the sequence H_{closed}. Both odd and even harmonics of H_{cone} have mathematically transformed into only the odd harmonics H_{closed} where the fundamental is an octave lower. The next section will explicitly show how this remarkable transformation occurs by graphing the numerical solutions as a function of B.

The general case for the conical frustum

Defining $x \equiv \pi\left(\frac{f}{f_0}\right)$ and $s \equiv \frac{B}{1-B} \geqslant 0$, equation (8) for the general conical frustum becomes

$$\tan x = -sx. \tag{13}$$

Equation (13) can be solved numerically by superimposing the plots for $y = \tan x$ and $y = -sx$. The parameter s is chosen for slope instead of the usual m to avoid confusion with m representing odd integers in this paper.

Many numerical solutions are necessary since for each B there is a slope s with its corresponding series of resonance frequencies. Figure 3 illustrates the method for the family of resonance frequencies given by equation (13). The overlapping points are the solutions, where the frequencies f are determined from the definition $x = \pi\left(\frac{f}{f_0}\right)$.

The conditions for a complete cone are $a = 0$, $B \equiv \frac{a}{b} = 0$, and $s \equiv \frac{B}{1-B} = 0$. The slope of the $y = -sx$ line shown in figure 3 must be zero for the complete cone. Therefore for this case, the linear graph is the horizontal line $y = 0$. The intersection points give the solutions $x \equiv \pi\left(\frac{f}{f_0}\right) = n\pi$, which is equivalent to $f_n = nf_0$. These solutions demonstrate that the complete cone of slant height L has resonance frequencies identical to the harmonic series for an open pipe of length L, which was noted earlier in equation (10).

The conditions for a closed cylindrical pipe are $a = b$, $B \equiv \frac{a}{b} = 1$, and $s \equiv \frac{B}{1-B} \to \infty$. Transforming the cone into a closed cylindrical pipe requires that the line $y = -sx$ swing downward approaching the vertical axis. This very steep line of negative slope intersects the tangent lines at $x \equiv \pi\left(\frac{f}{f_0}\right) = m\frac{\pi}{2}$, where m is odd, which is equivalent to $f_m = m\frac{f_0}{2}$. Therefore, the closed (cylindrical) pipe of length L has odd resonance frequencies with a fundamental an octave lower than an open (cylindrical) pipe of the same length. This observation was noted earlier in equation (12).

Visualizing the rotation of the line with zero slope to a line approaching negative infinite slope reveals how the solution of all harmonics transforms to odd harmonics with a fundamental reduced by an octave. A plot of $\frac{f}{f_0}$ as a function of B is given in figure 4 for the first eight resonances, revealing the transformation as one moves from $B = 0$ (complete cone) to $B = 1$ (closed cylindrical pipe). The values for figure 4 were obtained

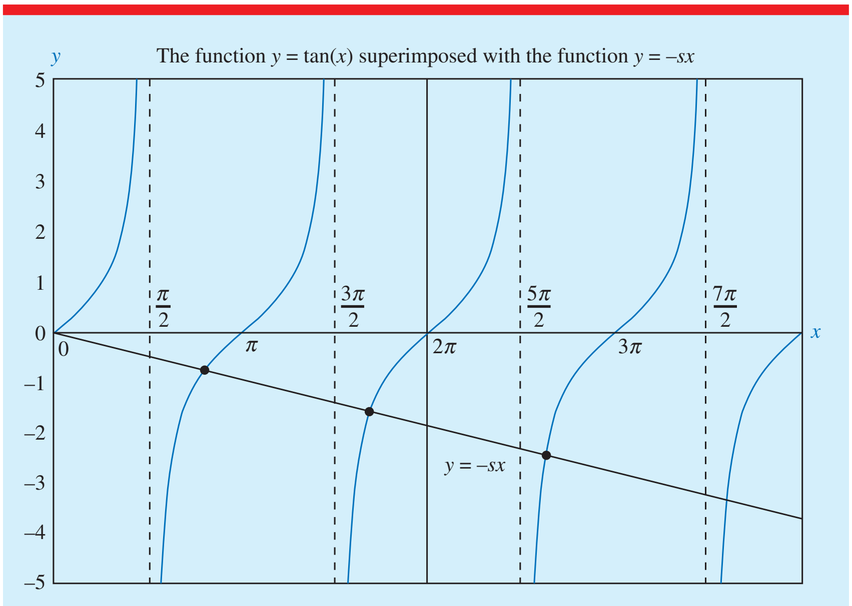

Figure 3. Solving the transcendental equation tan $x = -sx$ by superimposing the graphs $y = \tan x$ and $y = -sx$. The intersection points are the solutions for the values of x, from which the resonance frequencies are found: $f = \frac{x}{\pi} f_0$.

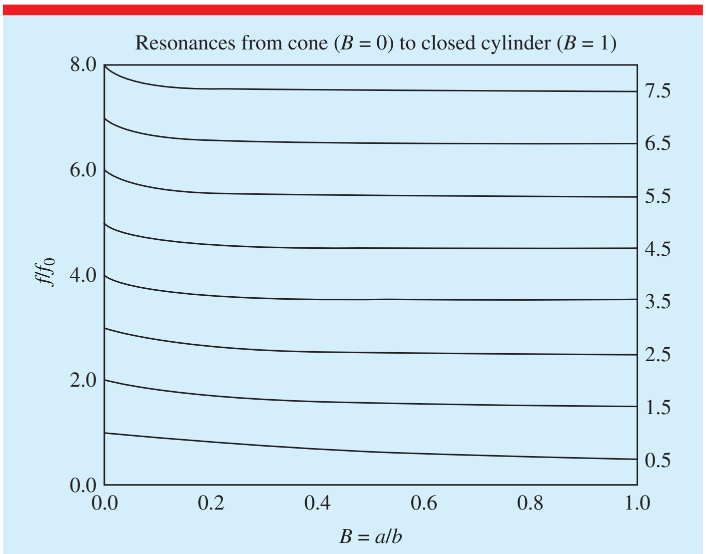

Figure 4. Plots of the first eight resonance frequencies of a conical frustum (slant length L and closed on the smaller end) relative to the fundamental frequency of an open pipe of length L. The value $B = 0$ indicates a complete cone and $B = 1$ defines a closed cylindrical pipe.

using *Mathematica* [3]. Note that the harmonics for the general conical frustum have non-integral ratios.

What about the cone angle? Earlier in equation (2) we gave an estimate on the restriction for the diameter d of a closed cylindrical pipe to be $d < \frac{L}{5}$. Therefore we want the larger diameter $b < \frac{L}{5}$. Let θ be the angle of the cone.

Then, $\sin\frac{\theta}{2} = \frac{b/2}{L} = \frac{1}{2}\frac{b}{L} < \frac{1}{2}\frac{1}{5} = \frac{1}{10}$ and $\frac{\theta}{2} < \sin^{-1}\frac{1}{10} = 5.7^\circ$, giving $\theta < 11^\circ$. The oboe, bassoon, saxophone, and tuba are among the instruments using conical bores. Though at first glance the maximum of 11° may seem small, musical instruments that use conical bores easily meet the criterion. Several instruments that employ conical bores are listed in table 1. Four woodwinds [4] and the tuba[2] are included. All cone angles satisfy the condition $\theta < 11^\circ$.

The HTML5 app: hearing the resonances

The author has written an HTML5 app so that readers and students can hear the first eight resonances transform from those of a complete cone to those of a closed cylindrical pipe [5]. A video abstract of this paper demonstrating the app and additional musical connections is provided [6]. The small end correction[3] for the open end is neglected since the diameter of the open end is not varied and this diameter can be considered to be very small compared to the length of the pipe. The emphasis instead is on the transformation of a complete cone into a closed cylinder as $0 \leqslant B \leqslant 1$, shown in figure 4. The user can turn on and off any one of the harmonics and scan the cases from a complete cone ($B = 0$) to a closed cylinder ($B = 1$). Students familiar with music can have fun comparing intervals and chords.

As an example, the interval between the first harmonic (H1) and second harmonic (H2) for the complete cone ($B = 0$) has a ratio H2:H1 = 2:1, an octave. By the time the user has scrolled to the closed cylinder ($B = 1$), the ratio is 1.5:0.5 = 3:1, an octave (2:1) plus a musical fifth (3:2) since $\frac{3}{1} = \left[\frac{2}{1}\right]\left[\frac{3}{2}\right]$. As another example, H3:H2 = 3:2 on the complete cone end and 2.5:1.5 = 5:3 on the closed cylinder end. Playing the H2 and H3 tones for the complete cone defines the beginning interval for the song 'Twinkle, Twinkle, Little Star.' The 5:3 interval for the closed cylinder is a perfect major 6th, the beginning of the song 'My Bonnie Lies Over the Ocean.'

Table 1. Conical bore angles for common musical instruments. All bore angles meet the $\theta < 11^\circ$ requirement.

Instrument	Conical bore angle
Oboe	1.4°
Bassoon	0.8°
Soprano saxophone	3.5°
Tenor saxophone	3.0°
F tuba	2.5°

Users can explore any combination of the eight resonances played simultaneously. Menu choices for a sample triad, tetrad, and pentad are included with some others. The app is rich in musical intervals, non-integral tunings, and physics.

Conclusion

The first eight resonances of a closed conical frustum have been analyzed and can be demonstrated with the use of the included HTML5 app [5]. The software consists of one code file along with a few image files that can be run online or downloaded in a combined zip file and run on a computer offline. The source code is found in the single code file documented with comments for computer programmers who would like to play with the code. The HTML5 app brings to life the solutions of the transcendental equations that emerge from the boundary conditions of the conical frustum.

An added profound pedagogical bonus can be extracted from this paper. Many years ago my undergraduate quantum mechanics teacher[4] told me that the secret back door entrance into quantum mechanics was through the discrete harmonics of strings and pipes. Students can be challenged to think of something in classical physics that requires specific energy levels or modes. See if they can come up with standing waves on vibrating strings or the tones of a pipe.

Modern physics texts show that equation (1*a*) in conjunction with de Broglie's relation $p = \frac{h}{\lambda}$ and $E = \frac{p^2}{2m}$ can be used to derive the exact quantum

[2] The F tuba result was found from estimating the radius at the end of the conical section that attaches to the flare to be 8 cm. The main conical section for the F tuba is 3.66 m. Therefore, the bore angle is $\theta = 2\sin^{-1}(8/366) = 2.5^\circ$.

[3] End corrections imply that $0.61r$ be added to the physical length L due to the open end where r is the radius at the open end. The pressure node extends a little beyond the open end of the pipe.

[4] J Richard Houston (1935–2011), Professor of Physics for over 50 years at St. Joseph's University, Philadelphia, PA, USA.

mechanical quantized energies for a particle of mass m in an infinite square well of width L [7]. Furthermore, the numerical approach to solving the transcendental equation for the conical frustum is the technique employed to find the bound-state energies for a particle in a finite square well [8]. Functions of the form $y = x \tan x$ and $x^2 + y^2 = R^2$ can be overlapped in the first quadrant, providing intersections similar to the points highlighted in figure 3 [9]. These solutions represent the bound states.[5]

In summary, the challenging problem of the closed-open conical frustum, where the closed end has the smaller diameter, has been presented in a basic way avoiding differential equations. The analysis reveals why all harmonics are relevant in musical instruments with conical bores (see table 1) and why the odd harmonics with a lowered fundamental (by an octave) apply to musical instruments with cylindrical closed pipes such as the clarinet. Finally, the approach in this paper leads the student through steps encountered in quantum mechanics, where boundary conditions lead to quantized energies. In our case, we find quantized frequencies. The HTML5 app allows one to hear these specific frequencies over the range of conical frusta from a complete cone to the closed cylinder.

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Appendix

The derivative of equation (7) can be worked out using the product rule $\frac{\mathrm{d}p(r)}{\mathrm{d}r} = \frac{\mathrm{d}(uv)}{\mathrm{d}r} = v\frac{\mathrm{d}u}{\mathrm{d}r} + u\frac{\mathrm{d}v}{\mathrm{d}r}$, where $u = \frac{1}{r}$ and $v = A\ \sin[k(r_0 + L - r)]$. The result is

$$\frac{\mathrm{d}p(r)}{\mathrm{d}r} = -\frac{A\ \sin[k(r_0 + L - r)]}{r^2} - k\frac{A\ \cos[k(r_0 + L - r)]}{r}. \tag{A.1}$$

Imposing the boundary condition at $r = r_0$,

$$-\frac{A\ \sin[k(r_0 + L - r_0)]}{r_0^2} - k\frac{A\ \cos[k(r_0 + L - r_0)]}{r_0} = 0. \tag{A.2}$$

Algebraic manipulation gives $\sin(kL) + kr_0\cos(kL) = 0$, $\frac{\sin(kL)}{\cos(kL)} + kr_0 = 0$, and finally

$$\tan(kL) + kr_0 = 0. \tag{A.3}$$

Using $k = \frac{2\pi}{\lambda}$ and $v = \lambda f$ leads to $k = \frac{2\pi f}{v}$. Let $f_0 = \frac{v}{2L}$, the fundamental for an open pipe of length L [2]. Then $v = 2Lf_0$, $k = \frac{2\pi f}{v} = \frac{2\pi f}{2Lf_0} = \frac{\pi f}{Lf_0}$ and $kL = \pi\frac{f}{f_0}$. Equation (A.3) becomes $\tan\left[\pi\left(\frac{f}{f_0}\right)\right] + kr_0 = 0$, which can written as

$$\tan\left[\pi\left(\frac{f}{f_0}\right)\right] = -\left[\pi\left(\frac{f}{f_0}\right)\right]\frac{r_0}{L}. \tag{A.4}$$

Define $B \equiv \frac{a}{b}$, the ratio of the smaller diameter to the larger diameter [2]. The introduction of this parameter enables consideration of the limiting cases where the pipe is a complete cone ($B = 0$ when $a = 0$) and a closed cylindrical pipe ($B = 1$ when $a = b$). From similar triangles in figure 2, $B = \frac{a}{b} = \frac{r_0}{r_0 + L}$. To obtain the explicit appearance of B in equation (A.4), we need to solve for $\frac{r_0}{L}$ in terms of B. The result is

$$\frac{r_0}{L} = \frac{B}{1 - B}. \tag{A.5}$$

Substituting equation (A.5) into equation (A.4) leads to

$$\tan\left[\pi\left(\frac{f}{f_0}\right)\right] = -\left[\frac{B}{1 - B}\right]\pi\left(\frac{f}{f_0}\right), \tag{A.6}$$

which is the desired equation (8) found earlier in the paper.

[5] Overlapping $y = x \tan x$ with the appropriate circle function in the first quadrant leads to the even wave-function solutions while using $y = -x \cot x$ with the circle gives the odd solutions. For another approach, see [7] where the parametrization does not use a circle but another function.

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Michael J Ruiz is professor of physics at the University of North Carolina at Asheville (UNCA), USA. He received his PhD in theoretical physics from the University of Maryland, USA. His innovative courses with a strong online component aimed at general students have been featured on CNN.