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Swinging beam of beads transforming into a simple pendulum of the same length

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Abstract

Costume jewelry beads are employed to demonstrate how a beam of beads is able to transform into a simple pendulum (single bead) where the pendulum length remains constant throughout. These beads are common and inexpensive for instructors, as well as attractive and exciting for students. Beads are removed from the top, keeping the same pendulum length until a single bead remains. The experimental data deviates about 2% from an ideal theoretical model for the extreme cases of the simple pendulum and rigid rod. Factors leading to the small discrepancies are discussed. This engaging experiment is demonstrated with an accompanying video (Hawes and Ruiz 2018 *Video: Fun with swinging beads* http://mjtruiz.com/ped/beads/).

Introduction

The pendulum is an important topic covered in introductory physics [1-2]. Papers continue to be written to help teachers communicate pendulum motion to their students in more effective and engaging ways [3-6]. For this paper, attractive and inexpensive jewelry beads were used to make three types of pendulums: a complete beam of beads on a string, a pendulum with a column of beads in only the lower portion of the string, and the simple pendulum (one bead attached at the end of the string). The total length of each type of pendulum was held constant during each experiment with a given set of beads. When doing a series of experiments with a specific type of beads, the fixed length was determined by how many beads of that type were available. See figure 1 for three kinds of beads.

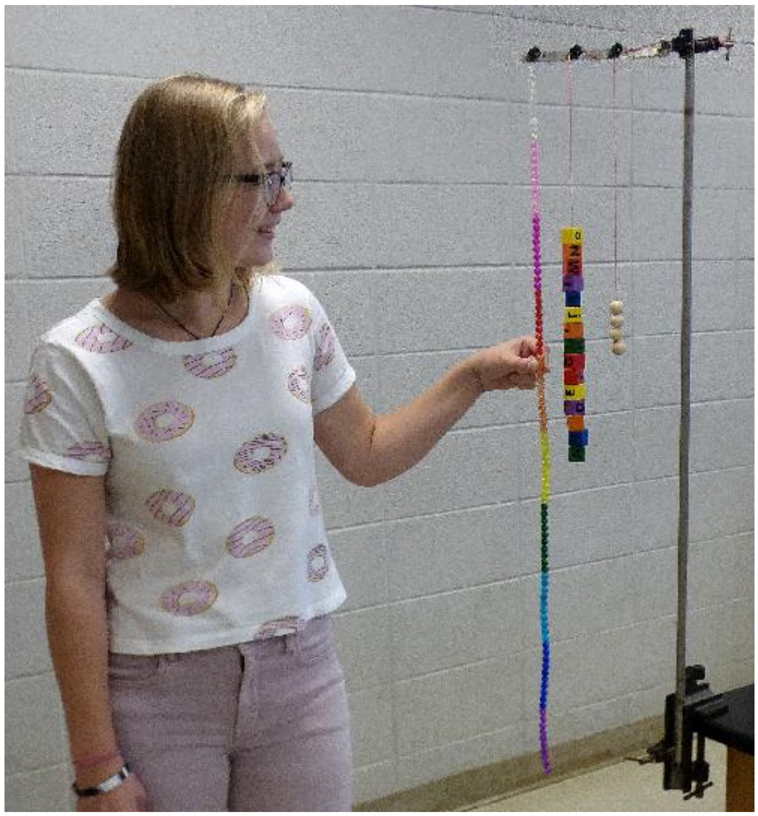

Figure 1. Coauthor Chloe Hawes with three types of beads. For each case, the experiment started with beads stacked from the bottom up to the pivot point with a very small space near the pivot to allow for free swinging. Beads were gradually removed from the top, keeping the total length of the pendulum constant for each series of experiments.

The left case in figure 1 is a beam of colored beads with the configuration for the beginning of the experiment. Beads were then removed a few at a time from the top until the last bead was reached - the simple pendulum. The pendulum was hung at the same point on the string each time so that the total pendulum length remained constant. The wooden blocks in the middle pendulum illustrate an experiment in progress for cubes, where some of the blocks have been removed from the top. For pendulum at the far right, removing three beads will result in the last measurement, determining the period of the simple pendulum.

The model

Students should understand the physical pendulum formula

$$T = 2\pi\sqrt{\frac{I_{pivot}}{M_{total}\, g L_{cm}}}, \quad (1)$$

where I_{pivot} is the moment of inertia about the pivot swinging point, M_{total} is the total mass of the pendulum, L_{cm} is the length from the pivot point to the centre of mass, and $g = 9.81\ \mathrm{m \cdot s^{-2}}$ is the acceleration due to gravity [2-4]. They should also know moments of inertia for some objects such as $I = ml^2$ for a point mass a distance l from the pivot, $I_{cm} = \frac{2}{5}mr^2$ for a sphere (uniform mass m, radius r) about its centre of mass [7-8], and $I_{beam} = \frac{1}{3}ml^2$ for a rigid beam (uniform mass m and length l) swinging by its end [2,4]. Finally, students need to know the parallel axis theorem, also referred to as Steiner's theorem, in order to calculate the moment of inertia for an object displaced by a distance d from its centre of mass. The parallel axis theorem is given by

$$I = I_{cm} + md^2, \quad (2)$$

where m is the mass of the object, I_{cm} is the moment of inertia about its centre of mass, and d is the displacement of the object from its centre of mass [8]. The parallel axis theorem provides an easy way to calculate the moment of inertia I about an arbitrary axis located a distance d from its centre of mass.

The model used for our experiment assumes rigid connections between the beads. See figure 2 for a diagram showing three beads. Taking the beads to be spheres with uniform mass m and radius r, the moment of inertia of each bead about its centre of mass is $I_{cm} = \frac{2}{5}mr^2$. For the cubes $I_{cm} = \varepsilon mr^2$ may be used where ε is a coefficient less than 1, m is the mass of the cube, and $2r$ is the side of the cube. As will be seen, the precise shape of the small bead hardly affects the theoretical predictions.

The parallel axis theorem is employed to find the moment of inertia for each bead from the pivot point. The moments of inertia are summed to obtain the total moment of inertia I_{pivot}. The moment of inertia is written down in equation (3) for n beads using figure 2 as a guide.

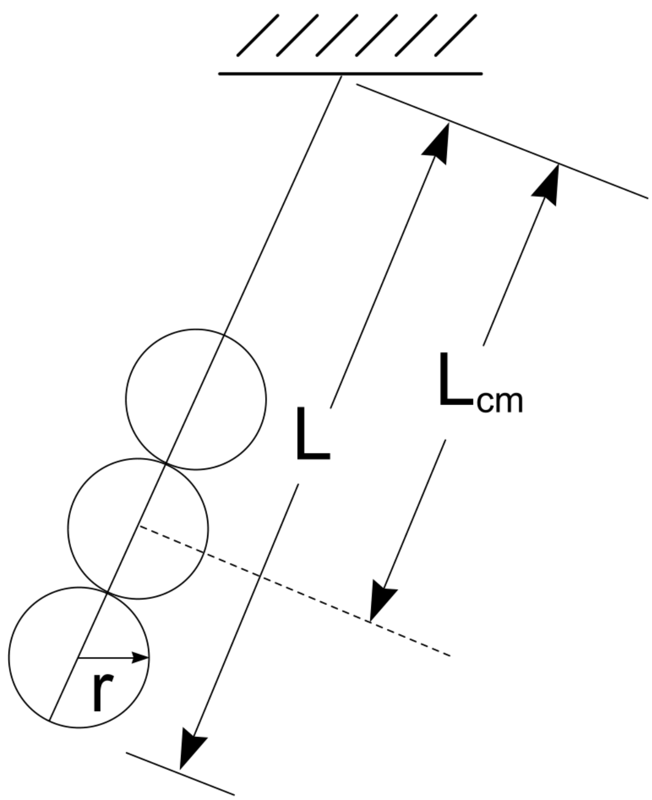

Figure 2. Model for beads forming a physical pendulum where beads are in the lower section of the pendulum and no beads are in the upper section.

The total moment of inertia for n beads is

$$
\begin{aligned}
I_{pivot} = & \frac{2}{5}mr^2 + m(L-r)^2 \\
& + \frac{2}{5}mr^2 + m(L-3r)^2 \\
& + \frac{2}{5}mr^2 + m(L-5r)^2 \\
& \cdot \\
& \cdot \\
& \cdot \\
& + \frac{2}{5}mr^2 + m[L-(2n-1)r]^2.
\end{aligned} \tag{3}
$$

Rearranging the terms leads to

$$I_{pivot} = n\frac{2}{5}mr^2 + mnL^2 - 2mrL[1+3+5+\ldots(2n-1)] + mr^2[1^2+3^2+5^2+\ldots(2n-1)^2]\,. \tag{4}$$

Note that no calculus is needed for this theoretical derivation. Rather, students have a chance to work with the mathematics of series [9]. The first series is the sum of the first n odd integers [10],

$$1+3+5+\ldots(2n-1) = n^2\,. \tag{5}$$

The second series is the sum of the first n odd integers squared [10],

$$1^2+3^2+5^2+\ldots(2n-1)^2 = \frac{n(2n+1)(2n-1)}{3}\,. \tag{6}$$

The distance from the pivot to the centre of mass is $L_{cm} = L - nr$. The total length L is equal to the maximum number of beads N (that reach to the very top of the string) multiplied by the diameter of a single bead: $L = 2rN$. The total mass of the beads is given by $M = nm$. Substituting these relations into the physical pendulum formula equation (1) leads to

$$T = 2\pi\sqrt{\frac{\frac{2}{5}Mr^2 + ML^2 - 2MrLn + \frac{1}{3}Mr^2(2n+1)(2n-1)}{Mg(L-nr)}}\,, \tag{7}$$

which simplifies to

$$T = 2\pi\sqrt{\frac{\frac{2}{5}r^2 + L^2 - 2rLn + \frac{1}{3}r^2(4n^2 - 1)}{g(L - nr)}} \quad . \tag{8}$$

To check equation (8) for the limiting case of the simple pendulum, set $n = 1$. The result is

$$T = 2\pi\sqrt{\frac{2r^2}{5g(L - r)} + \frac{(L - r)}{g}} \, . \tag{9}$$

Since $L = 2Nr = 200r \gg r$, the first term under the radical sign can be neglected and equation (9) reduces to the case for the simple pendulum

$$T = 2\pi\sqrt{\frac{l}{g}} \, , \tag{10}$$

where $l = L - r$, the length from the pivot to the centre of the single bead at the end of the string.

To check equation (8) at the other extreme, set $n = N$, the maximum number of beads. Equation (8) then becomes

$$T = 2\pi\sqrt{\frac{\frac{2}{5}r^2 + L^2 - 2rLN + \frac{1}{3}r^2(4N^2 - 1)}{g(L - Nr)}} \, . \tag{11}$$

Using $L = 2Nr$, neglecting the relatively very small $\frac{2}{5}r^2$, and taking $4N^2 - 1$ to be $4N^2$, the result is

$$T \approx 2\pi\sqrt{\frac{L^2 - L(2rN) + \frac{1}{3}(2r)^2 N^2}{g(L - Nr)}} = 2\pi\sqrt{\frac{L^2 - L^2 + \frac{1}{3}L^2}{g(L - L/2)}} = 2\pi\sqrt{\frac{2L}{3g}} \, , \tag{12}$$

which is the period for a rigid beam of length L [2,4]. The experiment reveals the transformation of a pendulum beam into the simple pendulum as beads are gradually removed.

The experiment

To start the experiment, the beads were strung on a string still attached to the spool. The chosen string was of a light colour, so a mark could be made at the top of the beads. If enough beads were available, they were strung to a length as close as possible to a meter. See figure 3 for photos of the preparation with two options in securing the last bead.

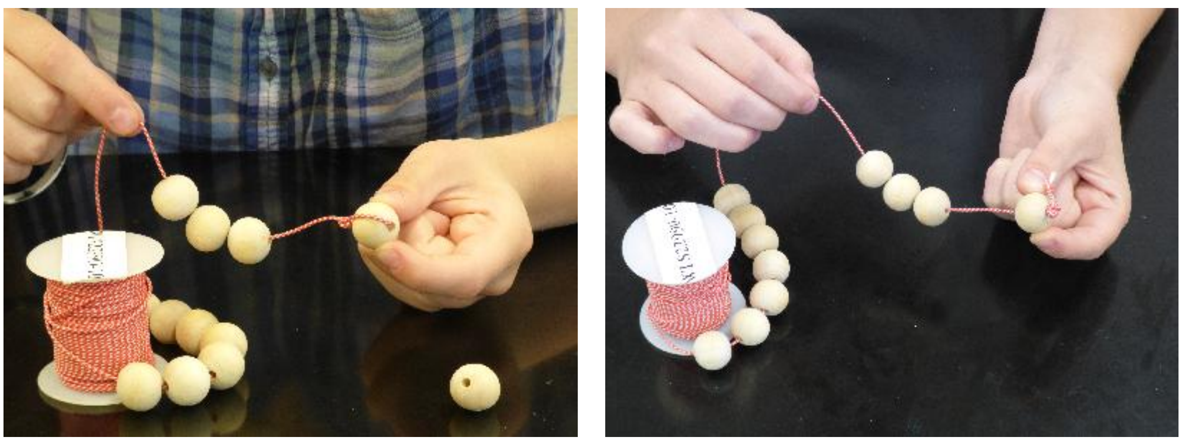

Figure 3. (a) The string attached to the last bead with a loop. Figure (b) The string attached to the last bead with a knot tied on the bottom.

The string was then hung at a point with a coin's width between the top bead and the pivot point of attachment. The top of the string was marked as close as possible to the attachment point to ensure that the string was replaced at the same place each time, after the process of removing beads. Two important parameters are the constant total length for the pendulum and the average diameter of each bead. The total length of the pendulum was measured with a meter stick. The average diameter of a bead was found by subtracting the small coin width from the pendulum length and dividing by the total number of beads.

The bottom of the pendulum was pulled to a small angle. Small angles of oscillation are important since then the pendulum can be considered to undergo simple harmonic motion, which is assumed in the derivation of the physics formulas. At times wave motion was observed along the vertical extent of the pendulum when there were many beads. The waves were due to the flexibility of the stacked bead array. Measurement of the period was postponed until these waves decreased significantly. In a later section, differences between the ideal rigid beam model and flexible string of beads will be addressed. Time was measured for 20 complete swings and this time was divided by the number of swings to arrive at the period for the oscillation. When timing, it is helpful to sway or move with the pendulum to better anticipate the end of each cycle and therefore stop the timer more precisely at the end of the swings. The period was recorded and then a group of beads was removed from the string. The experiment was then repeated until there was one bead left on the string. The data was entered into a spreadsheet along with the ideal theoretical results for the rigid bead array. A comparison between experimental and rigid-bead theoretical values is given in the next section.

The results

The results for the small colored beads appearing in figure 1 are plotted in figure 4 where the vertical axis is the period and the horizontal axis is the number of beads. In this experiment, the maximum number of beads was 105. This number of beads produced a complete beam that comprised the entire length of the pendulum. The first data point was taken with the period of 105 beads and 5 beads were removed from the top, leaving a gap in the upper section for the next pendulum. This process was continued until 5 beads were left. Then four beads were removed so that a single bead remained.

The theory based on rigid beads and the experimental values agree to within about 2%. The deviations occur at the two extreme regions of the graph. On the left extreme, the single bead in the simple pendulum has a very light mass so that the mass of the string cannot be neglected. The right side of the graph incurs deviations due to the flexibility of the beads along the string. With many beads, the string of beads must be considered as a hanging chain rather than a rigid beam or rod. Similar deviations as shown in figure 4 were obtained with the wooden beads and blocks. A detailed discussion of the discrepancies is presented in the next section.

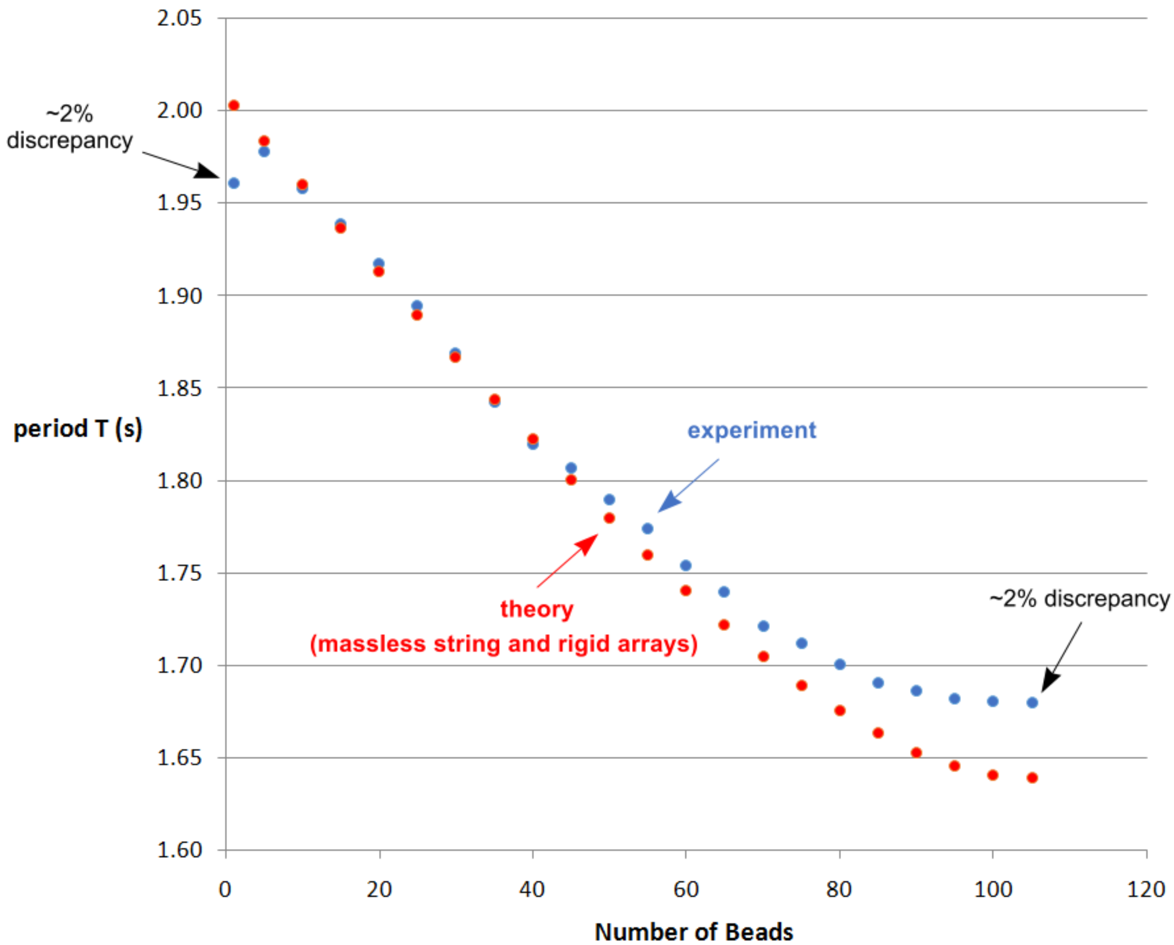

Figure 4. The experimental data compared to theory based on a massless string and rigid bead arrays. The 2% discrepancy for the simple pendulum arises since the string mass is not negligible for one bead. The 2% deviation for 105 beads is due to the flexibility of the beam.

Explaining the discrepancies at the extremes

The simple pendulum (one bead)

The measured value for the simple pendulum was 1.96 s, which is 2% lower when compared to the theoretical $T_{ideal} = 2\pi\sqrt{\frac{l}{g}} = 2\pi\sqrt{\frac{0.997\ \mathrm{m}}{9.81\ \mathrm{m}\cdot s^{-1}}} = 2.00\ \mathrm{s}$, where $l = L - r = 1.002\ \mathrm{m} - 0.0048\ \mathrm{m} = 0.997\ \mathrm{m}$. The discrepancy is there because the bead is so light that the mass of the string cannot be neglected. A single coloured bead has a mass of $m = 1.26\ \mathrm{g}$ and the string of length $L = 1\ \mathrm{m}$ used has a mass $\mu = 0.31\ \mathrm{g}$, giving the relationship $m = 4\mu$. When the mass of the string cannot be neglected, one has a stick and ball pendulum. The formula for the stick and ball pendulum is [4]

$$T_{bs} = 2\pi\sqrt{\frac{ml^2 + \frac{1}{3}\mu l^2}{g(ml + \frac{1}{2}\mu l)}}\ . \tag{13}$$

With $m = 4\mu$, equation (13) becomes

$$T_{bs} = 2\pi\sqrt{\frac{(4+\frac{1}{3})l}{(4+\frac{1}{2})g}} = 2\pi\sqrt{\frac{13}{3}\frac{2}{9}\frac{l}{g}} = 2\pi\sqrt{\frac{l}{g}}\sqrt{\frac{26}{27}}\ , \tag{14}$$

which shortens the period by the factor $\sqrt{\frac{26}{27}}$. Therefore, this factor should be applied to the theoretical 2.00 s, giving $2.00\cdot\sqrt{\frac{26}{27}} = 1.96\ \mathrm{s}$, in agreement with experiment. The next data point corresponds to 5 beads so that the bead mass now is 20 times that of the string. Therefore, agreement is quite good. Experimenting with different types of beads, the discrepancy at the simple pendulum end of the graph depends on the relative masses of the string and single bead.

The full beam (all beads)

For the systematic deviations appearing at the other extreme, reaching a 2% error for the full beam of beads, the error is due to the flexibility of the beam. A flexible beam is better described as a hanging chain. Lamb [11] indicates that to obtain the "gravest period" for a hanging chain of length l one must replace the rigid beam result

$$T_{beam} = 2\pi\sqrt{\frac{2l}{3g}} = 5.130\sqrt{\frac{l}{g}} \tag{14}$$

with the formula

$$T_{chain} = 5.225\sqrt{\frac{l}{g}} \ . \tag{15}$$

The period is therefore increased by the factor $\frac{5.225}{5.130} = 1.02$, which gives the 2% increase observed and evident in the graph of figure 4. Karls [12] obtains an increase in period of 2.3% measuring the period for a lamp ball chain with small spacings between the beads. The discrepancy for the last bead in figure 4 is more precisely 2.4%, matching very well the result of Karls. In either case, the error is about 2%, in agreement with the historical formula given by Lamb, which can be traced back to D. Bernoulli [11]. Note that as more and more beads are removed from the upper portion of the pendulum, the experimental data agrees better with the rigid-bead model until there is one bead remaining. Experimenting with different beads, the discrepancy at the full chain-of-beads end when compared to the rigid rod prediction is still 2%, as predicted by D. Bernoulli. Equation (15), the hanging-chain formula, must then be used instead of the rigid-rod formula of equation (14). These equations predict values that deviate by 2%.

Conclusion

The pendulum undergoing small oscillations is an excellent example of simple harmonic motion.

The pendulum is one of the most basic and popular applications encountered in a textbook chapter on oscillations. The experiment provided in this paper presents an opportunity for students to explore both the simple pendulum and physical pendulum in a laboratory setting. Colourful toy beads and blocks make the lab more attractive to students. Featonby has pointed out that toys "can often give an initial stimulus not possessed by the often sterile and remote special apparatus we use in the physics laboratory." [13] Beads and blocks are also easily viewed as real-world components by the students.

By starting with a beam of beads and transitioning to the simple pendulum, students can see how one limiting case can transform into another. A video abstract has been prepared describing the experiment [14]. The experimental data agrees within about 2% with a theoretical model based on rigid-bead arrays. These deviations present an opportunity for students to explore the deeper physics involving string mass and the hanging chain described earlier in this paper. It is easy for students to notice some vertical wave motion of the bead array when the maximum number of beads are used. This observation implies that the beam is flexible and similar to a hanging chain. After completing the lab students can turn the beads on the string into a necklace or bracelet.

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